THE

MATHEMATICAL GAZETTE

EDITED BY

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CHANGING VIEWS OF GEOMETRY

PRESIDENTIAL ADDRESS TO THE MATHEMATICAL ASSOCIATION, 14th APRIL, 1955.

BY W. V. D. HODGE.

I should like to preface this address with a word of thanks for the honour which the Association has done me in electing me as its President for the past year. The custom of the Association of choosing its Presidents in turn from those engaged in teaching in schools, and those whose main business is either teaching or research in our Universities, is a very healthy one, since it reminds us that we are all members of the same profession, and are all dependent on one another. The custom, however, has one serious disadvantage from the point of view of the Association, for it not infrequently results in the choice of a president whose other commitments leave him with insufficient time to devote to the affairs of the Association, however anxious he may be to fulfil his obligations. I fear that in this respect I have failed to give the Association the services I should have wished; but the Association has been well served by its other officers, and I should like to take this opportunity of thanking them, both for the way in which they have carried many of the burdens which should have been borne by the president, and for their continued and devoted services to the Association.

In view of the interest which the Association has always taken in the teaching of geometry, it is inevitable that when a professor of geometry occupies the presidential chair his presidential address should concern itself with some geometrical topic, but the choice of the actual topic is not quite so easy, I do not feel in the least qualified to speak on problems of teaching geometry, since the pupils who come my way have already undergone a lengthy process of instruction and selection, and hence present quite different problems from those which confront most of you. I am, of course, greatly affected by the results of all this preliminary training, and I may have something useful to say about the ends which you should have in view in training young mathematicians; but I am quite sure that I am wise in leaving the methods to be adopted to you, who have so much more experience in the problems of teaching school

children than I have. The most essential thing for those of us who have to train young mathematicians for research is that they should have a genuine love of their subject, and a real understanding of what it is about, and for this we are entirely dependent on the schoolmaster ; for with very few exceptions if a young man or woman has not acquired a genuine enthusiasm for mathematics before he comes to the University he is unlikely to acquire it at all.

I therefore think that I should try to give you some idea of what geometers are nowadays interested in, and for many reasons this seems an appropriate time to consider this. An appropriate starting point for an address on this subject to the Mathematical Association is provided by Professor Hardy's Presidential Address of 1925 on " What is geometry? " which is, I think, a very fair statement of what geometry is about, or at least was about when it was written. Geometers have every reason to be grateful to Professor Hardy for his fairness, for in 1925 geometry was going through a bad period, when many geometers were themselves confused about their objectives, and some other mathematicians were prepared to throw it out of the mathematical curriculum altogether. This was never Hardy's attitude, and I well remember that when I became Professor at Cambridge in 1936 he took great pains to help me to encourage developments in geometry along the lines that seemed best to me, and gave me much advice which proved of the greatest value.

The last thirty years have seen an enormous improvement in the position of geometry as a branch of mathematics, or, rather, have seen the re-integration of geometry into the main fabric of mathematics. Indeed, one can go further and say that with the restoration of geometry to its rightful place in the mathematical scheme the process of fragmentation which had been doing so much harm to mathematics has been reversed, and we may look forward to the day in which there are no longer analysts, algebraists, geometers and so on, but simply mathematicians. Mathematical research has two aspects, motivation and technique, and when the latter gains control the result is apt to be excessive specialisation. The revolution of geometrical thought, and the reinstatement of geometry as one of the major mathematical disciplines, have helped to bring about a unification of mathematics which we may justly regard as one of the major contributions of the last quarter century to the subject.

In 1925, geometry was suffering from a temporary exhaustion. The great nineteenth century geometers had created the vast edifice of projective geometry and had shown its relation to the older Euclidean geometry, and the general concept of geometry had been codified in Klein's Erlanger Program. According to Klein's point of view, a geometry is a system of definitions and theorems which express properties invariant under a group of transformations. Thus Euclidean geometry is concerned with invariants of the group of Euclidean movements, and projective geometry deals with the invariants of the group of collineations. Obviously, all the theorems of the geometry corresponding to a given group continue to be theorems in the geometry corresponding to any sub-group of the given group ; and the more restricted the group the more figures will be distinct relatively to it ; and the more theorems will appear in the geometry. The extreme case is the group corresponding to the identity, the geometry of which is too large to be of consequence.

This concept of geometry dominated mathematics for over half a century; but it was open to several serious objections. It reduced geometrical research to a series of exercises, and, as the more obviously interesting exercises became exhausted, geometers found themselves more and more pursuing investigations which interested fewer and fewer mathematicians, so that geometry, instead of being a major mathematical discipline, became an extreme specialisation. Again, the programme did not embrace all that was called geometry even when it was announced ; the geometry of Riemann's Inaugural Dissertation was

excluded, and it is probable that the reduction of Differential Geometry to a mere exercise in Differential Equations was at least in part due to the influence of Klein. The most significant advance in geometry between the Erlanger Program and 1925 was the development of birational geometry, particularly the theory of surfaces which we owe to the Italian school. In this theory we are not primarily interested in the properties of a surface as a locus in projective space, but only in the properties of all surfaces which can be transformed into one another by birational transformations ; the surfaces F and Φ given by the equations

$$F(x, y, z, t)=0, \quad \Phi(\xi, \eta, \zeta, \tau)=0$$

in the three-dimensional projective spaces (x, y, z, t) and (ξ, η, ζ, τ) are birationally equivalent if the points of F are traced out by the points (x, y, z, t), where

$$x=l(\xi, \eta, \zeta, \tau), \quad y=m(\xi, \eta, \zeta, \tau), \quad z=n(\xi, \eta, \zeta, \tau), \quad t=p(\xi, \eta, \zeta, \tau),$$

as (ξ, η, ζ, τ) describes Φ, and l, m, n, p are homogeneous polynomials of the same degree in (ξ, η, ζ, τ), and the points of Φ are similarly traced by the points (ξ, η, ζ, τ), where

$$\xi=\lambda(x, y, z, t), \quad \eta=\mu(x, y, z, t), \quad \zeta=\nu(x, y, z, t), \quad \tau=\pi(x, y, z, t),$$

as (x, y, z, t) describes F, λ, μ, ν, π being homogeneous polynomials of the same degree.

This again did not fall within the Erlanger Program, and one of the great debts which we owe to the founders of birational geometry is that they contributed more than anyone else towards breaking the artificial bounds set by it. At the same time, the Program had considerable influence on them, since it led them to develop the theory of curves and surfaces by seeking to find suitable representations of the loci under consideration in a projective space and using projective arguments to prove many of their theorems, a method which, as the subject advanced, became more and more difficult to apply.

Thus it appears to me that the authority with which Klein's definition has come to be regarded has had a cramping effect on geometrical thought, and that as a result geometers have pursued a narrow path which has led them to the isolated position in which they have found themselves until recently. At the same time, I ought to mention that the Kleinian concept of geometry plays an important role in the important advances made in the last thirty years by the late Professor Elie Cartan (which have the closest relation with the central theories of current mathematics), and while I think that the influence of Klein's ideas has been harmful to geometry, he did, at least, unite into a single concept an important group of geometries, and the fault lies more at the door of his followers than at Klein's.

The tyranny of the Erlanger Program has not yet finally disappeared, but I am prepared to say that its days are numbered. But, you will say, if the Erlanger Program, with its definition of geometry, is overthrown, what is geometry? It is much easier to say what geometry is not, than what it is, as Hardy found in his Presidential Address. To say, with Veblen, that we call a body of mathematical theorems a geometry because the name seems good seems to me to adopt an attitude of despair. The description of a geometry which I would hazard may at first sight seem to be a mere juggling with words, but it can be given substance. I should describe a geometry as the study of a space with a structure. The term *space* has been given a quite precise meaning by the topologists : it is an undefined aggregate of elements, called points, with an aggregate of sub-sets, called open sets, such that every

point of the aggregate belongs to at least one open set. Continuity can be defined in terms of open sets. For instance, the open sets of the Euclidean plane E_2 may be taken to be the sets of points of E_2 which have the property that if P belongs to an open set O_2 every point of E_2 distant from P by less than δ, for some positive δ, belongs to O_2. A function f in E_2 simply gives a mapping of E_2 on to the line E_1, where we can define open sets similarly. If f maps P of E_2 on Q of E_1, the function is continuous at P if, given any open set O_1 of E_1 containing Q, there exists an open set O_2 of E_2 containing P, such that the image of O_2 lies in O_1. In general, the open sets are arbitrary, and subject only to certain mild conditions; in particular it is often (but not always) required that if P and Q are distinct points, there exist open sets O and O', containing P and Q, respectively, which have no point in common.

A *structure* is imposed by giving properties of the open sets, and the relation between these properties in the intersection of open sets. This method of definition has been in use for a considerable time now in respect of differential geometry.

To define a differential manifold M, the kind of space considered in differential geometry, we impose the condition on the basic open sets (from which all the others can be found simply by adding sets together) that they should be sets which can be put in (1-1) correspondence with the interior of a sphere in Euclidean n-space, and that the images of open sets in this spherical region are open sets on M. This enables us to introduce a coordinate system (the coordinates of the images in the Euclidean space) into any basic region, and the differential structure of M is imposed by requiring the functions which give the transformation of coordinate systems in overlapping basic open sets to have continuous derivatives of a given order. Further structure conditions are required to define a Riemannian space; we must have in each basic coordinate system a quadratic differential form defining the square of the element of length at a point, with a suitable law relating the quadratic forms at a point common to two basic open sets. It is perhaps worth pointing out that these local properties are sufficient to enable us to develop differential geometry of the space, and it is not necessary to know much about the space as a whole, though at a more advanced level much current interest centres on the relation between local and global properties of a differentiable manifold.

This merely illustrates the method of defining a geometry by means of open sets and their structure. Professor Zariski has shown how elementary projective geometry, and the classical geometries of loci in projective space, can be usefully brought within this definition, and Professor Weil has carried his ideas further to give a more flexible definition of an algebraic locus.

It is, of course, true to say that a mere re-formulation of the definition of geometry has little effect on the day to day work of the ordinary geometer; but indirectly the effect may be enormous. If the geometer is led to reformulate his ideas in terms of the wider basic concepts, he may find that what he is doing, or trying to do, is akin to something which the analyst, say, who has likewise formulated his ideas in abstract terms, is doing, and the geometer and the analyst once more find themselves on speaking terms and each can help the other. This has, indeed, happened frequently in recent years, and as a consequence mathematicians have been led to concentrate their attention largely on a limited number of fundamental problems each of which has a bearing on a wide field of mathematics. This is what I meant when I said that the process of fragmentation of mathematics has been reversed.

At the same time I want to make it clear that I am not advocating that all geometries should be defined by means of open sets and structure conditions. Indeed, having pointed out some of the disadvantages that have arisen from the adoption of a rigid definition of geometry such as that due to Klein, I must at all costs avoid any risk of replacing it by a new definition. What I am

trying to do is simply to describe the features of a mathematical theory which are common to all theories to which one gives the name geometry for intuitive reasons, and I believe that all theories having these features have a common attraction for all those mathematicians whose natural way of thinking is usually described as " geometrical ".

In particular, I do not want to suggest any serious change in the way in which Euclidean or projective geometry is taught in schools. While a definition of, say, projective geometry is possible by means of open sets, I should regard such a definition as very clumsy, compared with many in common use. But I should like to say that in my opinion it is of the greatest importance that young students should understand that the concepts with which geometry deals are part of the whole fabric of mathematics, and that the difference between geometry and, say, algebra is really one of method. In this connection I believe that it is important to emphasise that algebraic and analytic methods in geometry are functions of the representation of a more abstract concept.

This brings me to the one point in Professor Hardy's address on which I consider his remarks were not entirely happy. You may recall that, after insisting that there is an infinite number of geometries, he divided them into two classes, the analytic geometries and the pure geometries, and he particularly stressed the fact that in analytic geometries, points, lines, etc. are *definite things*, and that there are no axioms, only definitions and theorems, whereas pure geometries are concerned with *all things that have certain properties* (the axioms). My criticism of Hardy's remarks on this subject is that it does not take sufficient account either of the concept of the representation of a geometry or the notion of equivalent geometries. I can perhaps best make clear what I mean by considering what Hardy calls common Cartesian geometry of two dimensions. This is the geometry in which the points are all the ordered pairs of real numbers (x, y) and the straight line is the aggregate of points satisfying a given linear equation $ax + by + c = 0$, where a, b, c are real numbers, and a and b are not both zero. If we now consider the pair of linear equations with real coefficients :

$$x' = px + qy + l, \qquad (ps - qr \neq 0)$$
$$y' = rx + sy + m,$$

these can be considered as defining a transformation of the plane taking the point (x, y) into the point (x', y'). It can then be *proved* that each point (x', y') arises from just one point (x, y), and that a straight line is transformed into a straight line. Now let us, abstractly, consider a set of elements S subject only to the condition that we can establish a (1-1) correspondence between its elements and the points of the (x, y) plane. This is simply the requirement that S should have the cardinal number of the continuum. We select one such correspondence. This enables us to attach to each element of S a pair of numbers (x, y) which we may call its coordinates. But the selected correspondence, combined with the transformation

$$x' = px + qy + l, \quad y' = rx + sy + m$$

defines another correspondence between the elements of S and the plane. Thus we are enabled to introduce allowable coordinate systems into S, and S with this set of allowable coordinate systems is an abstract notion of an affine space, and we could define the geometry by taking the open sets of S to be the images of open sets in the (x, y)-plane and the allowable coordinate system as the structure. The various coordinate systems give representations of the affine space as a common Cartesian geometry and in an obvious sense the various representations are all equivalent.

This method of defining a space and its structure by means of coordinate systems can be used to define (algebraic) projective spaces, and provides a perfectly satisfactory basis for the study of projective geometry. But, as you are aware, one can proceed to define projective spaces synthetically, by means of the propositions of incidence. The proposition of incidence (axioms) in the plane are as follows. There are two kinds of undefined entity, points and lines, and one property ; that of a point lying on a line (equivalently, of a line passing through a point). The following properties hold :

(1) there is exactly one line passing through two distinct points ;
(2) there is exactly one point lying on two lines.

Similar axioms are used for projective geometry of three dimensions, when we have three sets of independent elements, points, lines, and planes, and this idea can be carried on to higher dimensions. In the case of three dimensions it, can be *proved* (but unfortunately the proof is not easy) that we can introduce coordinates, so that a point is presented by a set of " numbers " (a, b, c, d) not all zero, and (a, b, c, d) and (a', b', c', d') represent the same point if and only if there exists a non-zero number λ such that $a' = \lambda a$, $b' = \lambda b$, $c' = \lambda c$, $d' = \lambda d$, where the " numbers " in question can be added, subtracted, and multiplied, and every number different from zero has an inverse a^{-1}, and that these numbers satisfy all the usual laws of algebra, except the commutative law of multiplication $ab = ba$. The commutative law holds if and only if the well-known theorem of Pappus holds, that is, if and only if given two sets of three points P, Q, R and P', Q', R' lying on two lines in a plane, the intersection $L = (QR', Q'R)$ $M = (RP', R'P)$, $N = (PQ', P'Q)$ are always collinear.

The case of two-dimensional plane geometry is slightly different. In the proof of this result for three-dimensional space, great use is made of the fact that Desargues' theorem on two triangles in perspective can be deduced from the proposition of incidence. In two dimensions this is not true, as can be shown by counter-examples, and we have to assume the truth of Desargues' Theorem as an additional axiom, and we can then proceed as in three dimensions. But if we want the commutative law of multiplication to hold, we have again to assume Pappus' Theorem, and it can be shown that the propositions of incidence plus Pappus' Theorem lead to a proof of Desargues' Theorem in two dimensions, so that if we want to ensure that the commutative law holds we do not require Desargues' Theorem as a separate axiom.

Even when we have the commutative law of multiplication holding, the coordinates of a point are not ordinary numbers, but belong to some field of rationality intrinsically associated with our basic set of elements (in special cases they may be ordinary numbers). This however does not matter for the point I wish to make. Given an abstract set of elements satisfying the propositions of incidence in three dimensions and Pappus' Theorem, the student has a choice of methods : he can either proceed to develop the geometry of the space directly from the axioms (the axiomatic method) or he can first introduce his coordinates and then investigate the geometry by algebraic means, using the appropriate field of rationality. It is quite clear that the results must be the same, and the method may be chosen as the student fancies. It is just a question of whether he likes axiomatic methods or representational ones, and personally I do not see why one method should be considered superior to another.

There are, of course, many ways of representing a geometry, and these are all equivalent in an obvious sense. I think this equivalence deserves to be brought out prominently, as it concentrates attention on the basic ideas common to all representations.

I have said that I do not care whether a geometer proceeds to work directly from axioms or prefers to work with an algebraic representation. In the best

of all possible worlds, one would like one's pupils to be equally expert in either method ; but this is asking too much in this imperfect world, and one can get along quite happily either way. I think that if a vote were taken, there would be an overwhelming majority in favour of the view that the representational method, by coordinates, was easier for the normal mathematician. I myself, however, am not entirely convinced of this, and I am encouraged to believe that many students could be led to appreciate the more axiomatic approach, which has a very strong aesthetic appeal, by an experience I had when I was a visiting professor at Harvard a few years ago. Much has been said about American standards of education, and I do not wish to say any more on that topic : while I think that there may be certain things to be said for their system, the fact remains that students enter their universities much less well prepared than ours do. It was therefore quite a shock to me, when I was asked to give a course on plane projective geometry, to be told that I should treat it axiomatically. Against my better judgment I agreed, and when I began the course I was quite astounded by the ready way in which the class took to the subject. Indeed, I should have been greatly pleased had my tripos candidates in Cambridge done as well. However, towards the end of the course I foresaw that it would be distinctly heavy going, and resolved to pass over to the representational method before I came to the most difficult topics. The switch over went quite smoothly, and I had no difficulty in convincing the class that a point could be represented by a set of homogeneous coordinates (x, y, z), and a straight line by a linear equation. The rest seemed easy : they accepted the fact that to find the point common to the lines.

$$ax + by + cz = 0, \quad \text{and} \quad a'x + b'y + c'z = 0,$$

they merely had to solve the linear equations. But when I gaily wrote down the solution $(bc' - b'c, ca' - c'a, ab' - a'b)$ I was greeted with cries of " How did you do that? ". The rest of the course, instead of being easier, turned out to be much harder.

The moral of this story seems to be that mathematical reasoning based on an axiomatic basis is not difficult, and the traditional British preference for algebraic methods is merely the result of the intensive drilling in algebra and algebraic methods which our students undergo. Modern mathematics is turning more and more to axiomatic methods, and there is a great deal to be said for introducing students to them at an early stage, when, I believe, they would take to them more readily, and enjoy them. For this purpose, axiomatic projective geometry is ideal.

However that may be, the main point of this address has been an attempt to convey to you some idea of the attitude of mind of the modern geometer, and if I have succeeded in communicating one or two ideas which you can pass on to your pupils I shall have succeeded in my task. In conclusion, may I say that, whatever the authors of G.C.E. or scholarship papers may want, what I, as a professor of geometry, want to see in young men entering the university is not any great ingenuity in solving tricky problems, but a thorough understanding of what geometry is about, and a sound, if elementary knowledge of the techniques required.

W. V. D. H.

THE MATHEMATICAL ASSOCIATION

The Annual General Meeting of the Mathematical Association was held at University College, Leicester, on 13th, 14th, 15th and 16th April, 1955. On 13th April members were welcomed by the Principal, Mr. C. H. Wilson, at a reception held in Beaumont Hall.

On 14th April, the business meeting was held at 9.15 a.m., with the President, Professor W. V. D. Hodge, in the Chair. The Report of the Council for 1954 was adopted, and a financial statement given by the Treasurer was received. The election of Mr. G. L. Parsons as President for 1955–6 was announced. The existing Vice-Presidents were re-elected, and Professor W. V. D. Hodge became a Vice-President under Rule 19. The Treasurer, the Secretaries, the Editor of the *Mathematical Gazette*, the Librarian and the Auditor were re-elected. The following were elected to serve on the Council: Dr. I. W. Busbridge, Miss W. M. Lehfeldt, Mr. E. H. Lockwood, Mr. H. V. Lowry, Dr. E. A. Maxwell, Mr. F. J. Phillips, Mr. M. A. Porter, Mr. A. P. Rollett, Miss K. M. Sowden.

An addition to Rule 8 was proposed by the Treasurer: "An Ordinary Member who has (i) retired from full-time work, and (ii) passed the age of 60, and (iii) paid at least 30 annual subscriptions, will be exempt from the payment of further annual subscriptions provided that he pays a final subscription of two guineas and indicates in writing to one of the Secretaries before the 31st January in each year his wish for his membership to be continued." This was carried.

At 9.45 a.m. Professor Hodge gave his Presidential address, "Changing views of geometry." The illness of Mrs. Neville prevented the attendance of Professor Neville to speak on the books in the Library of the Association, but Professor T. A. A. Broadbent spoke briefly on the origin and growth of the Library, and Mr. Prag gave a paper on "Mathematical Literature", with reference to some of the Association's books. In the afternoon, members visited places of interest in the Leicester area, and theatre parties were arranged for the evening.

On 15th April, at 9.15 a.m., Professor R. L. Goodstein spoke on "The definition of number"; at 10.15 a.m., a discussion on the recently issued Report of the Association on "The teaching of mathematics in technical colleges" was opened by Mr. F. W. Kellaway, Dr. H. Frazer and Mr. A. J. L. Avery. At 2 p.m. Mr. W. O. Storer opened a discussion on "The disadvantages of a mathematical education", and at 3.15 p.m. Mr. W. M. Wreathall spoke on "High speed computers and their application to lens design". At 5 p.m. Dr. Paul White gave a paper entitled "What is the matter with negative mass?" At 8 p.m. a social evening was held in the Hall of Wyggeston School.

On 16th April, at 9.15 a.m., Mr. M. J. Moroney spoke on "The application of statistics to industry".

A Publishers' Exhibition and an exhibition of apparatus used in the teaching of number in Infant Schools were open during the meeting, and many members also took the opportunity of inspecting the Association's Library, housed in the Library of University College, Leicester, where a number of specially interesting items were on display.

GLEANINGS FAR AND NEAR

1826. Poisonous Doctrine.

Borel considers that this paradox, which he would regard as an antimony, condemns the axiom of choice. This point of view is not shared by others.—*Mathematical Gazette*, December 1954, p. 262. [Per Mr. G. Nicholls.]

REPORT OF THE COUNCIL FOR THE YEAR 1954

Membership.

During the year ended 31st October, 1954, 196 ordinary members and 30 junior members were admitted to the Association. At the end of the year the membership figures were Honorary, 8; Ordinary, 2,285; Junior, 60; Life, 239; a total of 2,592 compared with 2,503 at the beginning of the year; the first increase for five years. The following are the membership totals since 1948: 1948, 2,570; 1949, 2,679; 1950, 2,643; 1951, 2,537; 1952, 2,506; 1953, 2,503; 1954, 2,592.

It is with regret that the Council reports the death of the following members: Professor A. A. K. Ayyangar (1936), Mr. S. G. Brown (1945), Mr. G. E. Crawford (1945), Mr. W. A. C. Curjel (1949), Miss W. Garner (1916), Mr. H. C. Jones (1952), Professor Gino Loria (Honorary).

Finance.

Income for the year ending 31st October, 1954, was £3,958 0s. 2d. and expenditure was £3,770 18s. 8d., giving an excess of income over expenditure of £187 1s. 6d. During last year, £649 13s. 6d. was spent on reprinting four of the major Reports; these reprints should provide a valuable source of income for this year. The new report on *The Use of Visual Methods* and the new *List of Books* together cost £296 8s. 10d., so that the printing and reprinting of Reports cost approximately £950.

The various items of income and expenditure have now become sufficiently stabilised for a fairly confident forecast to be made, and some £1,000 should be available for Library expenses and printing during the year ending 31st October, 1955. If this amount is so expended, very little will be left in hand for the following year. The Association still needs to increase its membership, and to have many more of its members helping very substantially by signing the 7-year covenant.

The Mathematical Gazette.

The need for economy keeps the *Gazette* down to about 320 pages per annum, though enough material of high standard is being offered to fill an annual volume at least one half as large again. In particular, the old-established policy of covering a very wide field in the review pages means that a considerable fraction of space is occupied by reviews; but opinions expressed by readers seem to indicate that this policy is one which is fully approved by members of the Association.

Library.

Accessions to the Library since the transfer to Leicester have been confined to periodicals exchanged for the *Gazette* and to numerous gifts listed in the *Gazette*.

During the session 1953–54, 132 books and periodicals were borrowed, of which 115 were sent by post, and in the first half of the present session 78 books and periodicals have been borrowed, excluding borrowing by members of the Leicester University College staff.

A start has been made in dealing with the immense arrears in binding and 65 volumes of periodicals have been bound.

The Teaching Committee.

The Teaching Committee, as elected at the last Annual General Meeting, met at King's College, London, on January 6th, 1954. The names of members of the Committee, and a report of that meeting, were circulated to members of the Association with the *Gazette* for May 1954.

The *Report on the Use of Visual Methods in Mathematics*, and the revised *List of Books suitable for School Libraries*, were issued to members with the *Gazette* for February 1954. The *Report on the teaching of Mathematics in Technical Colleges* is to be circulated with the February 1955 number of the *Gazette*.

The *Report on the Teaching of Mathematics in Primary Schools* has been completed and is in the hands of the printers. It is hoped to circulate this with the *Gazette* for December 1955. The sub-committee working on the *Teaching of Mathematics in Secondary Modern Schools* is making steady progress, but their Report will depend substantially on that of the Primary Schools sub-committee. Other sub-committees have been active, and when not meeting have been engaged in thought; there is something to be said for financial stringency as a regulator of output.

During 1954 it became necessary to reprint four Reports—*Arithmetic*, *Calculus*, *Mechanics* and *Second Geometry*. It was decided not to reprint the report on *Mathematics in Secondary Technical Schools*.

No meeting of the whole Teaching Committee has been held between the two Annual General Meetings; such a meeting would be expensive, and it is uncalled for so long as the various sub-committees are going on steadily with their respective tasks.

The Branches.

The work of the Branches has been continued with many interesting meetings which are reported periodically in the *Mathematical Gazette*. Most of the Home Branches have held discussions on the forthcoming report of the Secondary Modern sub-committee. In many cases these discussions have been opened by a member of the sub-committee, and Secondary Modern teachers who are not members of the Association have attended the meeting and expressed their opinions. This widening of the activities of the Association to include Secondary Modern School problems represents a welcome advance.

Problem Bureau.

There has been a substantial increase in the number of inquiries for solutions. These are, for the most part, routine requests which can be answered by sending solutions from the files. Demands for the solution of "puzzles" have almost disappeared, probably not from any lack of interest, but because there are now more books which give solutions to them. Thanks are again due to the team of solvers who assist in the work of the Bureau. It is regrettable that there is a shortage of interesting problems to pass on, with the exception of those from the Cambridge Scholarship papers. The Bureau could easily handle many more problems, and members should not hesitate to send them to Dr. G. A. Garreau, 90 Wyatt Park Road, London, S.W. 2.

Officers and Council.

Council wishes to offer, on behalf of all members, its sincere thanks to the President, Professor W. V. D. Hodge, and to the Officers for the work which they have done for the Association during the year.

THE ORIENTAL INFLUENCE ON GREEK MATHEMATICS

By R. J. Gillings.

The problem of finding rational integral numbers which could be made the sides of right triangles, is said by Proclus to have been solved by Pythagoras, by a method which is equivalent to the use of the formula,

$$m^2 + [\tfrac{1}{2}(m^2 - 1)]^2 = [\tfrac{1}{2}(m^2 + 1)]^2$$

where m is any odd integer.

Heath (1) quotes Proclus' statement as follows :

" The method starts from odd numbers. For it makes the odd number the smaller of the sides about the right angle ; then it takes the square of it, subtracts unity, and makes half the difference the greater of the sides about the right angle ; lastly it adds unity to this and so forms the remaining side the hypotenuse."

Another method attributed by Proclus to Plato depends on the formula

$$n^2 + [(\tfrac{1}{2}n)^2 - 1]^2 = [(\tfrac{1}{2}n)^2 + 1]^2$$

where n is any even number.

Again quoting Heath's (1) rendering of Proclus :

"But the method of Plato argues from even numbers for it takes the given even number and makes it one of the sides about the right angle : then bisecting this number and squaring the half, it adds unity to the square to form the hypotenuse, and subtracts unity from the square to form the other side about the right angle."

Coolidge (2) gives the Platonic formula as,

$$(2n)^2 + (n^2 - 1)^2 = (n^2 + 1)^2$$

remarking that there are few strictly mathematical passages in Plato's works, and that while Plato was certainly familiar with the Pythagorean theorem, " the only direct mention of it is in the special case where we have an isosceles right triangle, which appears at length in Meno 82–4."

How did the Greeks arrive at these formulae or methods?

From the sources, it is by no means clear to what extent the equivalent of what we may call theoretical algebra had developed with the Pythagoreans. Various conjectures, plausible enough no doubt, have been offered, as for example, Bretschneider (Heath 1 p. 358) (1), that Pythagoras had deduced his method inductively by observing the array of numbers,

	1	2	3	4	5	6	7	8	9	10	11	12	13...
Their squares	1	4	9	16	25	36	49	64	81	100	121	144	169...
Their differences		3	5	7	9	11	13	15	17	19	21	23	25...

and noting which numbers in the difference row are perfect squares. Or again, Treutlein's suggestion (Heath 1 p. 358), (1) is that it came from the consideration of gnomons, and the arrays of dots commonly referred to as " figurate " numbers,

0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0

where in the diagram, it is clear that for this case, the gnomon is a square number, and thus $4^2 + 3^2 = 5^2$. Similar explanations have also been made to

explain the source of the Platonic formula. Both are deducible as particular cases from Euclid's proposition 28 lemma 1 of Book X of his *Elements*, which may be written as

$$p^2q^2 + [\tfrac{1}{2}(p^2 - q^2)]^2 = [\tfrac{1}{2}(p^2 + q^2)]^2,$$

the proof of which is given by Euclid in geometrical form. It has also been maintained that Pythagoras obtained his theory from India, (Bürk, Heath, p. 360), (1) where the theorem was said to have been known and proved in its generality, as early as the sixth century B.C.

However, the extent to which Greek mathematics was indebted to the work of the Babylonians over 1000 years earlier has yet to be adequately determined, and the problem of how the Babylonians determined these Pythagorean triads may shed some light on the question. A Babylonian clay tablet written in cuneiform, (−1900 to −1600) now in the Plimpton collection of Columbia University N.Y., referred to as Plimpton 322, contains a list of 15 triads, some of which are large and unusual numbers. The list is as follows, where l, b and d are connected by the relation $l^2 + b^2 = d^2$.

line	l	b	d
1	120	119	169
2	3456	3367	4825
3	4800	4601	6649
4	13500	12709	18541
5	72	65	97
6	360	319	481
7	2700	2291	3721
8	960	799	1249
9	600	481	769
10	6480	4961	8161
11	60	45	75
12	2400	1679	2929
13	240	161	289
14	2700	1771	3229
15	90	56	106

So far, no procedure texts have been found which give an indication of the method by which the Babylonian scribe arrived at these numbers, and the interest of the historians of mathematics has been excited to determine just how the Babylonian scribe accomplished it. Now, in addition to the numbers above, the tablet has a column of numbers giving the values of d^2/l^2 for each line, which suggests that the Babylonian Mathematician was concerned to list his triads in a certain order, an order which was determined by a regular changing of the shapes of the right triangles to which the numbers referred. By itself, this does no more than adumbrate the theory behind the method. But there are further indications which arise from the accidental circumstance that the scribe who wrote the tablet committed four errors. An examination of these errors is very enlightening. The first is a mere scribal error, where an extra wedge was inadvertently put in, a 9 written for an 8. The second error is the writing of 53 instead of 1,46 (106 in decimal notation), the scribe having omitted to multiply by 2, and the third error is the writing of 7, 12, 1 (25, 921 in decimal notation), instead of 2,41 (161 in decimal notation), that is, the scribe wrote the square of the number instead of the number itself. Thus to calculate the numbers of the tablet, the doubling of numbers, and the recording of squares, from their abundant table texts, must have been part of the

procedure. But the committal of the fourth error in line 2, where the scribe wrote 3, 12, 1 (11,521 in decimal notation), for 1, 20, 25 (4,825 in decimal notation), has given students of Babylonian mathematics much food for thought, and an intelligible explanation of how the scribe came to make this error must shed light on the ultimate algebraic theory by which the Pythagorean triads were determined. Error number four is nothing like as simple as the other two errors. Neugebauer had written of it (3):

"It seems to me that his error should be explicable as a direct consequence of the formation of the numbers of the text. This should be the final test for any hypothesis advanced to explain the underlying theory."

The explanation or conjecture which follows here, and which has withstood examination by Neugebauer himself, shows that it is probable that the scribe was using the equivalent of the formula,

$$[2pq]^2+[p^2-q^2]^2=[p^2+q^2]^2$$

where p and q are integers prime to one another, are not simultaneously odd, and p is greater than q.

For the numbers of the second line the scribe chose $p=1,4$ (64 in decimal notation), and $q=27$, and the values for l and b in the equation, $l^2+b^2=d^2$, become,

$$\begin{aligned} l=2pq &= 2\times 1,\ 4\times 27 \\ &= 57,\ 36 \\ &= 3456 \text{ in decimal notation.} \\ b=p^2-q^2 &= (1,\ 4)^2-27^2 \\ &= 1,\ 8,\ 16-12,\ 9 \\ &= 56,\ 7 \\ &= 3367 \text{ in decimal notation.} \end{aligned}$$

For d, the substitutions give,

$$\begin{aligned} d=p^2+q^2 &= (1,\ 4)^2+27 \\ &= 1,\ 8,\ 16+12,\ 9 \\ &= 1,\ 20,\ 25 \\ &= 4825 \text{ in decimal notation. (4)} \end{aligned}$$

Thus the scribe obtained the primitive triad 3456, 3367, 4825; truly a remarkable performance, over 1000 years before Pythagoras is said to have celebrated the discovery of the similar relation between the numbers 3, 4 and 5, by the roasting of an ox.

The explanation of the error (4) is that the scribe, in determining the value of $d=p^2+q^2$, reversed the order of squaring and adding and thus evaluated $(p+q)^2$ instead of p^2+q^2. Thus he would have obtained $(1,\ 4+27)^2=(1,\ 31)^2$ $=2,18,1$ merely by reading from his table of squares. Having made this very commonplace error, (as has been repeated by many a careless schoolboy since), he would have corrected it by subtracting $2pq$, which he should have determined by evaluating,

$$2(1,\ 4)\times 27,$$
$$\text{or } 2(1,\ 0+4)\times 27,$$
$$\text{or } 54(1,\ 0+4).$$

Then he adds, $54\times 1,\ 0=54,\ 0$ (omitting the portion of the product 54×4, by

error), and writes the sum,

$$\begin{array}{r} 2,\ 18,\ 1 \\ 54,\ 0 \\ \hline 3,\ 12,\ 1 \end{array}$$

The omission of the 54×4, as part of the scribe's mistake is the suggestion of O. Neugebauer, and is plausible enough as suggested by other errors found in extant texts. Until another explanation of the commission of the error of writing 3, 12, 1 for 1, 20, 25, by the scribe is discovered, which is more plausible than the foregoing, we may accept it as explaining the underlying theory, and it leads us to accept, what is already pretty well indicated by Plimpton 322, namely, the fact that the Babylonian scribe was familiar with the algebraic determination of the integral numbers representing the three sides of right angled triangles, in the millenium before the Greeks provided a theoretical proof of the relation. Is it beyond the bounds of reasonable possibility that the Babylonians transmitted their mathematical knowledge to the Greeks? The following passage from Heath's *Mathematics in Aristotle* (Oxford, 1949, page 18), refers to the Babylonians as the " barbarians ".

" Eclipses had been observed ' by certain barbarians ', but the knowledge of the eclipse came to the Greeks later, through Thales. This is of course an allusion to the story that Thales predicted a solar eclipse which took place during a battle between the Lydians and the Medes, (probably the eclipse of May 585 B.C.) The explanation of the prediction is no doubt that Thales had learnt directly or indirectly from the Babylonians, the period of 223 lunations after which eclipses recur, which period had been discovered by the Babylonians as a result of observations of eclipses through long centuries. (It is curious that in the passage in Simplicius about ' certain barbarians ', it is eclipses of the moon which are spoken of as having been known to them, but unknown to the Greeks before Thales)."

And Neugebauer says of the Oriental influence of scientific Greek Mathematics (3), (p. 141),

" The theory of irrational quantities and the related theory of integration are of purely Greek origin, but the contents of the ' geometrical algebra ' utilize results known in Mesopotamia." And he goes on to say that the traditional stories of discoveries made by Thales or Pythagoras must be discarded as totally unhistorical. As time goes by, and more mathematical texts are translated, one does not doubt that the debt which the Greeks owed to the Babylonian culture of over 1000 years earlier will be shown to be greater than has hitherto been thought. And the emphasis given by certain historians, to the Egyptian influence on the origins of Greek geometry, from the necessity of land measurement resulting from the annual flooding of the Nile river, should shift to the algebraic and arithmetic influence of the earlier and far superior mathematics of the Babylonians, as revealed by their cuneiform clay tablets.

R. J. G.

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WHEN IS A PARALLELOGRAM NOT A PARALLELOGRAM?

BY P. GANT

1. Is a quadrilateral in which one pair of opposite sides are equal and one pair of opposite angles are equal necessarily a parallelogram? This question arose in class recently and led me to discover much that I did not know about an elementary topic. Possibly others may be interested in the results of my investigation.

2. The following 'proof', for which I am indebted to my colleague, E. H. Lockwood, answers the question in the affirmative. The subsequent paragraph contradicts this.

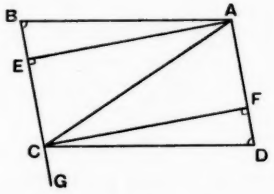
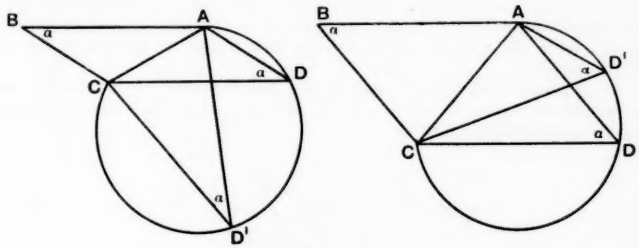


FIG. 1.

Given $AB=CD$ and $\angle B=\angle D$, we prove that $ABCD$ is a parallelogram as follows.

Drop perpendiculars AE, CF (Fig. 1), and produce BC to G. The triangles ABE, CDF are congruent (A.A.S) and hence $AE=CF$. But the quadrilateral $AECF$ is cyclic, since $\angle E+\angle F=180°$; so $\angle ACE=\angle CAF$, being subtended by equal chords AE, CF. Thus BC, AD are parallel and the alternate angles D and DCG are equal. But by hypothesis, $\angle B=\angle D$, and so $\angle B=\angle DCG$. Thus AB is parallel to CD, and $ABCD$ is a parallelogram.

3. Given $AB=CD$ and $\angle B=\angle D=\alpha$ (Figs. 2, 3) a quadrilateral may be constructed as follows.



FIGS. 2, 3.

Draw AB and take any point C such that $\angle ABC=\alpha$. On AC and on the side opposite to B construct an arc containing the angle α. With centre C and radius AB draw a circle. Since a parallelogram is always possible, the circle cuts the arc once at D, making $ABCD$ a parallelogram. Hence the circle centre C either cuts the arc again at D', giving a second quadrilateral $ABCD'$ or it cuts the complementary arc on the opposite side of AC (Fig. 4).

In the latter case, however, $\angle AD'C$ is reflex, and not equal to $\angle B$, so that the quadrilateral $ABCD'$ does not satisfy the given conditions. This case arises when the circle centre C and radius AB cuts AC between A and C, that is, when $AC > AB$. Thus a second quadrilateral $ABCD'$ is possible if $AC < AB$. For this to occur, $\angle B < \angle ACB$ and α is necessarily an acute angle.

To sum up, if $\alpha \geqslant 90°$, only a parallelogram $ABCD$ is possible. If $\alpha < 90°$, and if $AC < AB$, a second quadrilateral $ABCD'$ is possible.

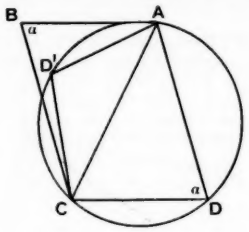


FIG. 4.

4. If $\alpha < 90°$, the condition $AC < AB$ is equivalent to choosing C between B and X where $AB = AX$, and $\angle B = \angle X = \alpha$ (Fig. 5). If C is taken on BX produced, only a parallelogram $ABCD$ is possible.

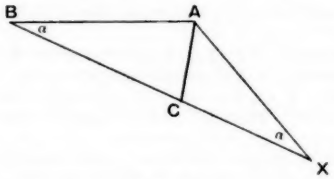


FIG. 5.

Since $\angle X = \alpha$, the circle on AC containing the angle α on the side opposite to B passes through X. It can thus be more conveniently constructed as the circumcircle of the triangle ACX. Note that for all choices of C the circles ACX are coaxal.

5. As appears from Figs. 2 and 6, the quadrilateral $ABCD'$ may have a reflex angle at either C or A. It is of interest to find the condition for a convex quadrilateral $ABCD'$.

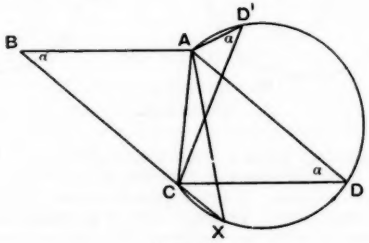


FIG. 6.

Note that the triangles CAD, CAD' (Figs. 2, 3, 6) satisfy the conditions for the ambiguous case. For, AC is common, $CD = CD' = AB$, and

$$\angle D = \angle D' = \alpha.$$

Provided $AC < AB$ the shorter given side is opposite the given angle and there are two non-congruent triangles. In this case the angles CAD and CAD' opposite to the longer given side are supplementary, by the ambiguous case.

Thus
$$\begin{aligned}\angle BCD' &= \angle BCA + \angle ACD' \\ &= \angle BCA + (180^\circ - \angle CAD' - \alpha) \\ &= \angle BCA + \angle CAD - \alpha \\ &= 2\angle BCA - \alpha.\end{aligned}$$

Hence $\angle BCD' < 180^\circ$ provided $\angle BCA < 90^\circ + \frac{1}{2}\alpha$.

Also, considering the angles of the quadrilateral $ABCD'$,

$$\begin{aligned}\angle BAD' &= 360^\circ - \angle BCD' - 2\alpha \\ &= 360^\circ - 2\angle BCA - \alpha.\end{aligned}$$

Therefore $\angle BAD' < 180^\circ$ provided $\angle BCA > 90^\circ - \frac{1}{2}\alpha$.

Hence for a convex quadrilateral $ABCD'$ we need

(i) $AC < AB$;

and (ii) $90^\circ - \frac{1}{2}\alpha < \angle BCA < 90^\circ + \frac{1}{2}\alpha$.

Note that (ii) is equivalent to

(iii) $90^\circ - \frac{3}{2}\alpha < \angle BAC < 90^\circ - \frac{1}{2}\alpha$.

6. The extreme positions of C for a convex quadrilateral $ABCD'$ are C_1 and C_2 (Fig. 7), where $\angle AC_1B = 90^\circ - \frac{1}{2}\alpha$ and $\angle AC_2B = 90^\circ + \frac{1}{2}\alpha$.

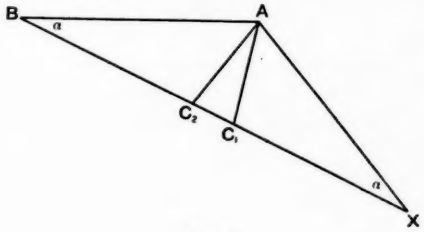


FIG. 7.

Hence $\angle BAC_1 = 90^\circ - \frac{1}{2}\alpha = \angle BC_1A$,

and so $BC_1 = AB$.

Also, $\angle AC_2C_1 = 90^\circ - \frac{1}{2}\alpha = \angle AC_1C_2$,

and so $AC_1 = AC_2$.

Thus $XC_2 = BC_1 = AX = AB$.

The figure can thus be constructed simply by ruler and compass. If C is chosen between C_1 and C_2, and the circle ACX and the circle centre C and radius AB are drawn, a convex quadrilateral $ABCD'$ is obtained. Note that

$\angle BAC_2 = 90° - \frac{3}{2}\alpha$. Hence the above figure is correct only if $\alpha < 60°$. If $\alpha = 60°$, C_1 is at X and C_2 at B, and C may be chosen at any point of BX. If $\alpha > 60°$, C_1 is on BX produced and C_2 on XB produced ; C may be chosen at any point between B and X.

7. The relative chances of getting a convex quadrilateral $ABCD'$ for different values of α may be estimated by the ratio of the length of the segment along which C may be chosen to the length of AB; that is, by C_1C_2/AB for $0 < \alpha \leqslant 60°$, and by BX/AB for $60° \leqslant \alpha < 90°$.

Now (Fig. 7),

$$C_1C_2 + BX = BC_1 + C_2X,$$

or

$$C_1C_2 + 2AB\cos\alpha = 2AB,$$

that is,

$$C_1C_2/AB = 2(1 - \cos\alpha).$$

This increases from 0 to 1 as α increases from 0° to 60°.

Again,

$$BX = 2AB\cos\alpha$$

or

$$BX/AB = 2\cos\alpha.$$

This decreases from 1 to 0 as α increases from 60° to 90°.

Thus the best chance of a convex quadrilateral $ABCD'$ occurs when $\alpha = 60°$.

8. The locus of D (completing the parallelogram $ABCD$) as C varies along BX is the line through A parallel to BX (Fig. 8).

The locus of D' is not so simple. When C is at C_2, $C_2X = AB$ and the circle centre C_2 and radius AB cuts the circle AC_2X at X. Hence X is on the locus of D'. For this extreme quadrilateral ABC_2X, the angle at C is 180°. Similarly, it is easily seen that the point Y on BA produced such that $C_1Y = AB = C_1B$ (and hence such that $\angle Y = \alpha$) corresponds to the other extreme position C_1. In this case the angle at A is 180°.

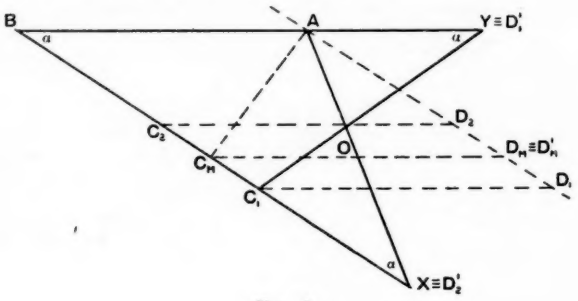


FIG. 8.

When C is the midpoint C_M of both C_1C_2 and BX, $\angle AC_MX = 90°$, since $AB = AX$, and hence AX is a diameter of the circle AC_MX. By the midpoint theorem, O is the midpoint of AX and hence the centre of this circle. Hence the centres C_M, O of the two circles are collinear with one intersection D_M of the circles. Hence the circles touch at D_M, and D_M' coincides with D_M.

Thus as C moves from C_2 to C_1, D' moves from X to Y via D_M.

9. To find the curve of which the locus of D' discussed in the last paragraph is part, we now let C move along the infinite line BX without restriction and find where the circle ACX cuts the circle centre C and radius AB. We

disregard whether or not the quadrilateral $ABCD'$ satisfies the given conditions and we disregard the point D which completes the parallelogram in each case. It is easily seen that as C moves from C_1 to X, D' moves from Y to A (Fig. 6), and that as C moves along BX produced to a point C_3 such that $XC_3 = XC_2 = AB$, D' returns to X again.

The locus can be sketched by drawing the perpendicular bisector of AX, taking different points on it as centres in turn, finding the point C for each circle ACX and then drawing the circle centre C and radius AB.

10. The locus of D' is obtained analytically as follows. Take X as origin, XB as x-axis, B as the point $(2, 0)$, A as the point $(1, m)$, where $m = \tan \alpha$. Let C $(\lambda, 0)$ be any point on XB (Fig. 9).

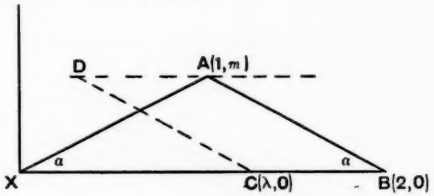


FIG. 9.

Any circle cutting XB at X and C is of the form

$$x(x-\lambda)+y^2+2fy=0.$$

Making $A(1, m)$ lie on this, we obtain as the equation of the circle ACX,

$$mx(x-\lambda)+my^2+y(\lambda-1-m^2)=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

The circle centre C and radius AB is

$$(x-\lambda)^2+y^2=1+m^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Rewriting (1) as

$$(x-\lambda)(y-mx)=y(my+x-1-m^2),$$

we may eliminate λ and so obtain

$$(1+m^2-y^2)(y-mx)^2=y^2(my+x-1-m^2)^2.$$

This fourth degree locus is the locus of intersections of the circles (1) and (2); that is, it contains the combined loci of D and D'. But the locus of D is the line $y=m$ through A parallel to BX. Removing the factor $y-m$ we obtain as the locus of D' the nodal cubic

$$y^3+x^2y+m(x^2-y^2)-2xy=0.$$

It is easily shown that this has an asymptote $y=-m$ parallel to BX through the reflection A' of A in BX; and that the tangents at the node X are perpendicular to each other.

P. GANT.

1827. The proportion of university students to the rest of the population is more than twice as high as it is in Britain, whereas the chances of an Irish graduate being absorbed into a profession unless he emigrates are less than half as good as the Englishman's.—*The Observer*, December 19, 1954. [Per Dr. S. Vajda.]

THE BOUNDARY LAYER AND "SEAM" BOWLING

By J. C. Cooke

Introduction.

I have never seen any explanation of why it is possible for a fast bowler to make a new cricket ball swerve, and why it becomes more difficult or impossible to do this when the shine has worn off the ball. Of course a sliced golf ball swerves because of the spin it has, and this swerve is a well-known illustration of the Magnus effect. Spin cannot be the explanation in the case of fast swerve bowling, because if it were the effect would increase as the ball became older and rougher. In any case a fast bowler imparts little or no spin to the ball.

I present here a possible explanation of the phenomenon.

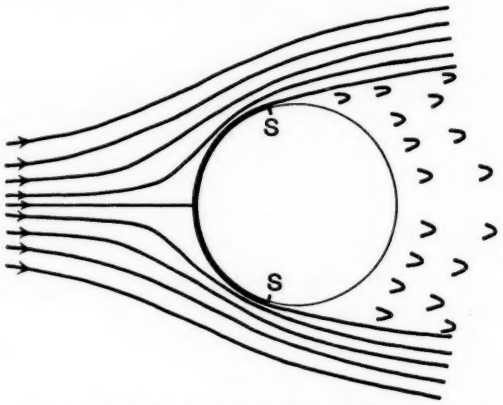


Fig. 1. Streamlines past a sphere. Laminar Separation.

The Boundary Layer.

We state first a few properties of the boundary layer as applied in particular to a sphere :

1. The boundary layer is a very thin layer close to the surface of the ball where the effects of friction or viscosity are important. Everywhere else the viscosity of the air can be neglected.

2. The boundary layer is thinner the faster the ball is travelling. It is also thinnest near the front.

3. The flow in the boundary layer is always laminar near the front of the ball, that is flowing evenly in layers, but as it travels round the surface it may become turbulent or wildly irregular. When it does this it thickens. Surface roughness assists transition to turbulence, especially a fairly large projection like the seam, but smaller projections or corrugations will do it, especially if they are near to the front where the boundary layer is extremely thin.

4. However, the boundary layer may separate from the surface before it has arrived at the turbulent regime. This separation is the same phenomenon as occurs when an aeroplane wing "stalls".

5. A laminar boundary layer separates more easily than a turbulent layer, that is to say it separates before it has been able to travel very far round the ball from the front. The point of separation is about 80° from the front. A turbulent layer is able to cling to the surface much longer, separating perhaps as much as 120° from the front.

6. There is a critical Reynolds number $2au/\nu$ (see below for the meanings of the symbols) above which the flow always becomes turbulent.

Property 5 is well illustrated by Figs. 1 and 2, which are drawings based on some photographs of Wieselsberger (2). These are reproduced in Goldstein (1) and in many other places. In Fig. 1 the layer is laminar and separates before the "equator" is reached. In Fig. 2 the boundary layer has been made

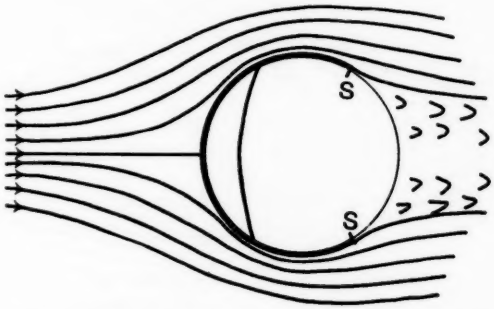


FIG. 2. Streamlines past a sphere. Turbulence induced by a wire hoop on the surface. Turbulent Separation.

turbulent artificially by a small wire hoop wrapped round the surface, and this enables the flow to travel further round the ball before it separates. The points of separation are marked S.

Effect on a Cricket Ball.

Consider now a cricket ball travelling without rotation with the seam at an angle to the direction of the motion, as shown in Fig. 3. The Reynolds number is supposed to be below the critical. The view is taken from above looking downwards.

At the top of the figure the surface is smooth and shiny and so the boundary layer is laminar and separates quite early as in Fig. 1. As the air travels round the part of the ball in the lower half of Fig. 3 it encounters the seam which makes the boundary layer turbulent. As a result of this the separation this side is delayed and occurs as in Fig. 2.

As a result of this difference in the two halves, the ball experiences a resultant force (in addition to the drag) in a direction indicated by the arrow, and so ceases to travel in a straight line and develops a swerve.

When the ball is worn and the shine is gone the boundary layer becomes turbulent even on the upper side of Fig. 3, owing to the roughness of the surface, and so the flow is then the same as in Fig. 2 and there is no tendency to swerve.

Some numerical values will be given later.

I realise that this " explanation " may well introduce more problems than it solves. For instance in Fig. 3 the frictional force on the surface of the ball at the lower half of the figure is probably greater than at the upper half. This would cause the ball to rotate anticlockwise. Does the bowler allow for this so that the seam is at the correct angle relative to the direction of flight at the moment he wishes the ball to swerve? A ball which swerves steadily from the time it leaves the bowler's hand is not difficult to play. What is required is for the swerve to take place as late as possible.

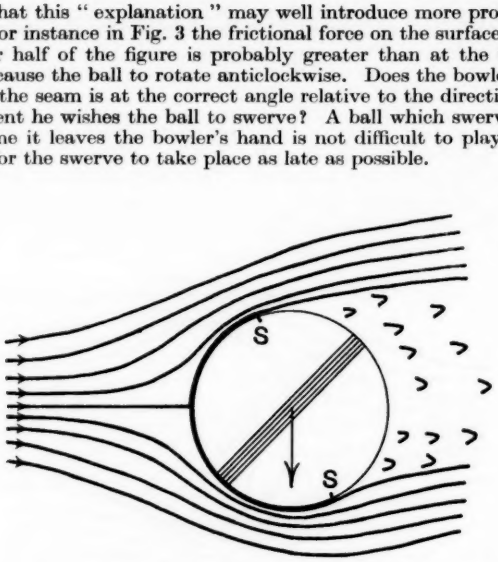


FIG. 3. Streamlines past a cricket ball. Top, laminar separation. Bottom, turbulent separation. Side force in the direction of the arrow.

This explanation applies to *fast* bowling. For slow speeds the boundary layer is so thick that surface roughness does not affect the flow to a very great extent. The approximations of boundary layer theory do not apply very well when the speed is low.

Some Numerical Results.

We shall consider the ball to have a radius (a) of 1·43 inches and a mass (m) of 5·5 ounces. I am told that Larwood was able to deliver the ball at a speed of 80 miles per hour, but that about 60 miles per hour is more usual. We shall take the speed (u) as 90 feet per second. The kinematic viscosity (ν) will be taken as 0·00016 feet^2/second and the density (ρ) as 0·075 pounds/cubic foot. These figures are taken from Goldstein (1) for a temperature of 20° C. We shall assume the ball is at rest and air flowing past it.

In this case the Reynolds number of the flow ($2au/\nu$) is about $1{\cdot}4 \times 10^5$ which is just below the critical number for change of flow from laminar to turbulent. We may make a crude estimate of the " cross wind " force in the following way. We take spherical polar coordinates r, θ, ϕ associated with Cartesian coordinates x, y, z in the usual way, and suppose that the air flow at infinity is parallel to the z-axis and of magnitude $-u$. The pressure normal to the sphere is denoted by p and the pressure far upstream (where the velocity is $-u$) by p_0. The components of the pressure on the sphere along the x-, y- and z-axes will be $-p \sin\theta \cos\phi$, $-p \sin\theta \sin\phi$ and $-p \cos\theta$ respectively. We make the assumption that for $0<\phi<\pi$ the regime is the laminar one, and for $\pi<\phi<2\pi$ it is turbulent. This will be approximately

true, except that there will not in fact be an abrupt discontinuity in p at $\phi=0$ and $\phi=\pi$. In each range p will be a function of θ only, and we shall take its magnitude from two curves of $(p-p_0)/\frac{1}{2}\rho u^2$ against θ given in Goldstein (1) in Figure 202 on page 497.

The laminar region will be represented by the curve given there for the lowest Reynolds number, and the turbulent region that given for the highest number. We write $p=p_1$ for $0<\phi<\pi$ and $p=p_2$ for $\pi<\phi<2\pi$, that, is for the laminar and turbulent regions respectively. The surface element being $a^2\sin\theta\,d\theta\,d\phi$, the three components of force on the sphere are equal to

$$-a^2\iint p\sin^2\theta\cos\phi\,d\theta\,d\phi, \quad -a^2\iint p\sin^2\theta\sin\phi\,d\theta\,d\phi,$$

$$-a^2\iint p\sin\theta\cos\theta\,d\theta\,d\phi,$$

the limits in each integral being $\theta=0$ to π, $\phi=0$ to 2π.

These reduce to

$$0, \quad -2a^2\int_0^\pi (p_1-p_2)\sin^2\theta\,d\theta\ (=L), \quad -\pi a^2\int_0^\pi (p_1+p_2)\sin\theta\cos\theta\,d\theta\ \ (=D).$$

In performing the integrations the values of p_1 and p_2 were estimated from the curves at 10° intervals and the integrations evaluated by Simpson's rule. The results are

$$L=0{\cdot}93a^2\tfrac{1}{2}\rho u^2, \quad D=0{\cdot}27\pi a^2\tfrac{1}{2}\rho u^2.$$

We may verify that the drag force D so obtained has a reasonable value since the drag coefficient C_D (given by $D=C_D\pi a^2\frac{1}{2}\rho u^2$) is equal to 0·27, and this is, as it should be, somewhere in between the drag coefficient for fully laminar and fully turbulent flows.

We can deduce the radius of curvature of the path from these results. The component of acceleration of the ball perpendicular to its direction of motion is L/m, and so in time δt the ball will have travelled a distance $\frac{1}{2}(L/m)(\delta t)^2\,(=Y)$ to the second order in δt parallel to the normal. Meanwhile it will have travelled a distance $u\delta t\,(=X)$ to the first order in δt parallel to the tangent. Hence the radius of curvature of the path, $\lim(X^2/2Y)$, will be equal to mu^2/L or $2m/0{\cdot}93\rho a^2$.

This gives a radius of curvature of 690 feet, a fairly large value, but nevertheless it represents a swerve of about 1 inch in 11 feet. Does a fast bowler make the ball swerve more than this?

It is supposed to be easier to make the ball swerve in a hot damp atmosphere. In such circumstances ν is larger than normal and so the Reynolds number is reduced. This means that it is easier to maintain the laminar part of the regime, as the layer, being thicker, is not so affected by surface roughness, yet is still thin enough to be affected by the seam. Hence shine is not so important and a fairly old ball can still be made to swerve.

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TWO THEOREMS ON ORDINARY NON-LINEAR DIFFERENTIAL EQUATIONS

By N. W. McLachlan

1. Theorem 1. In the differential equation

$$\ddot{y}+g(y)\dot{y}+f(y)=0, \quad \ldots\ldots(1)$$

if (i) $g(0)\geqslant 0$, $g(y)>0$ when $|y|>0$; (ii) $f(0)=0$, $f(y)$ is odd, bounded and >0 in $0<y\leqslant l$, while $\int_0^l f(y)\,dy\to+\infty$ as $l\to\pm\infty$; then for any non-zero initial condition(s) $y=y_0$, $\dot{y}=v=v_0$, both y and $\dot{y}\to 0$ as $t\to+\infty$.

Proof. Taking $v=dy/dt$, (1) becomes

$$v\,dv/dy+g(y)v+f(y)=0, \quad \ldots\ldots(2)$$

so

$$\int v\,dv+\int f(y)\,dy \quad =A-\int g(y)v^2\,dy, \quad \ldots\ldots(3)$$

A being the constant of integration. Since $dy=v\,dt$, we get

$$\tfrac{1}{2}v^2+\int_0^y f(y)\,dy=A-\int_0^t g(y)\,v^2\,dt=W, \text{ say.} \quad \ldots\ldots(4)$$

By (i) the t-integral is positive, while for the given initial conditions

$$A=\tfrac{1}{2}v_0^2+\int_0^{y_0} f(y)\,dy>0, \quad \ldots\ldots(5)$$

since $f(y)$ is odd. Also the left-hand side of (4) is never negative, so $0\leqslant W\leqslant A$. Then as $t\to+\infty$:

(a) if v were oscillatory, the t-integral in (4) would increase indefinitely, which is impossible, since $0\leqslant W\leqslant A$.
(b) if v tends to a limit other than zero, $y\to\infty$, which is impossible as in (a);
(c) if y tends to a limit, both $\dot{y}$ and $\ddot{y}\to 0$, so by (1) $f(y)\to 0$ and, therefore, by (ii), y must $\to 0$.

Hence both y and $\dot{y}\to 0$ as $t\to+\infty$.

2. *Example.* Imagine a mechanical system (or an analogue thereof) in which a massless spring is anchored at one end and fixed to a mass m at the other. The mass slides over a smooth horizontal plane and is constrained to have one degree of freedom. A damping device is connected to m which introduces a resistive force $r|y|\dot{y}$, thereby entailing loss of energy. When the displacement of m from its equilibrium position is y, the restoring force due to the spring is $f(y)=s_1y+s_3y^3$, where r, s_1, s_3 are positive constants. The equation for the motion of m is

$$m\ddot{y}+r\,|\,y\,|\,\dot{y}+s_1y+s_3y^3=0, \quad \ldots\ldots(1)$$

or

$$\ddot{y}+2k\,|\,y\,|\,\dot{y}+ay+by^3=0, \quad \ldots\ldots(2)$$

with $\kappa=r/2m$, $a=s_1/m$, $b=s_3/m$. Equation (2) may be written

$$dv/dy=v'=-(ay+by^3+2\kappa\,|\,y\,|\,v)/v. \quad \ldots\ldots(3)$$

When $y=v=0$, the r.h.s. has the indeterminate form 0/0, so the origin is a singular point (the only one). To determine the type of singularity we use Poincaré's criteria.* Equation (3) has the form

* *Jour. de Math.*, 8, 251, 1882 ; *Les Méthodes nouvelles de la mécanique céleste*, vol. 1, 1892 ; also E. L. Ince, *Ordinary Differential Equations*, p. 69, where the symbols a, b, c, d correspond to α, β, γ, δ in our case.

$$v' = [\alpha y + \beta v + P_1(y, v)]/[\gamma y + \delta v + Q_1(y, v)], \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

with $P_1(y, v) = -(by^3 + 2\kappa \mid y \mid v)$, $Q_1(y, v) = 0$. Thus $\alpha = -a$, $\beta = \gamma = 0$, $\delta = 1$,

so $$\left.\begin{aligned} \alpha\delta - \beta\gamma &= -a < 0, \\ (\beta - \gamma)^2 + 4\alpha\delta &= -4a < 0, \\ \text{and} \quad (\beta + \gamma) &= 0. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

These criteria show that the origin is either a focal or a spiral point, but presence of the higher order terms in $P_1(y, v)$ precludes discrimination between these two types. However, by theorem 1, it follows that y and $\dot{y} \to 0$ as $t \to +\infty$, so the origin is a stable spiral point. Consequently, the curve in the v-y plane, corresponding to any initial conditions $y = y_0$, $v = v_0$, spirals inwards about 0 with ever decreasing radius vector $(y^2 + v^2)^{1/2}$. Hence the motion is oscillatory whatever the value of $\kappa > 0$. If $\kappa = 0$, the v-y curves are closed about the origin, so it is a focal point, the motion then being periodic.

3. *Application to hydro-electric system.* If for y in (2), § 2, we write u and put $b = 0$, then

$$\ddot{u} + 2\kappa \mid u \mid \dot{u} + au = 0 \quad (a, \kappa > 0). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

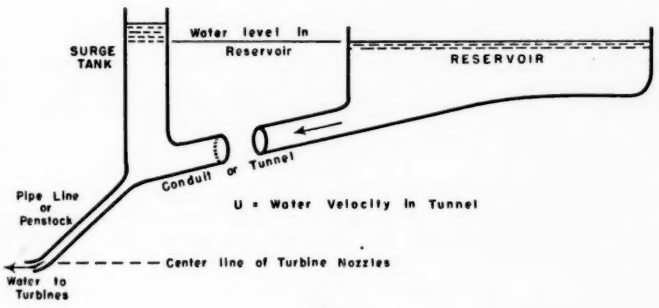


FIG.

This equation pertains to a hydro-electric power system (see Fig.). u is the velocity of the water when it oscillates in the tunnel connecting the reservoir with the turbines which drive the electric generating machines. At the lower end of the tunnel there is a large masonry tank open at the top to the air. The (surge) tank, tunnel and reservoir constitute, in effect, an enormous U tube filled with thousands of tons of water. When the sluice valves controlling the supply of water to the turbines are either opened or closed, the water in the U is set into oscillation. The function of the surge tank is to permit this oscillation, thereby reducing the force on the valves to a safe value. The kinetic energy of the water is dissipated in friction on the sides of the U, this being represented by the term $2\kappa \mid u \mid \dot{u}$ in (1). The term au is associated with the gravitational restoring force when the water level in the tank rises above or falls below its level for steady flow to the turbines. (1) pertains to the condition after all the valves have been closed. When one or more valves is opened, water flows down the tunnel to the turbines, and the equation is then

$$\ddot{u} + 2\kappa \mid u \mid \ddot{u} + au = d \quad (a, d, \kappa > 0). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

To investigate the motion corresponding to (2), we put $u = (x + c)$ and obtain

$$\dddot{x} + 2\kappa \mid x + c \mid \dot{x} + ax = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

with $c=d/a$. It follows from theorem 1 that x and $\dot{x}\to 0$ as $t\to+\infty$. Moreover, when $t>t_1$, $c>|x|$ and $|x+c|$ may be written $(x+c)$. Then with $v=\dot{x}=dx/dt$, $v'=dv/dx$, (3) gives

$$v'=-[ax+2\kappa(x+c)v]/v, \quad \ldots\ldots(4)$$

so $x=v=0$ is the only singular point. In other words $u=c$, $v=0$ is the only singular point of (2), this being the equilibrium position of the system; for when $\dot{u}$, $\ddot{u}$ are zero, the motion is unidirectional and $u=d/a=c$. In (4) $\alpha=-a$, $\beta=-2\kappa c$, $\gamma=0$, $\delta=1$, so

$$\left.\begin{aligned}\alpha\delta-\beta\gamma&=-a<0,\\(\beta-\gamma)^2+4\alpha\delta&=4(\kappa^2c^2-a),\\(\beta+\gamma)&=-2\kappa c<0.\end{aligned}\right\} \quad \ldots\ldots(5)$$

and

If $\kappa^2c^2\geqslant a$, the singularity is a stable node, so the motion is non-oscillatory. If $\kappa^2c^2<a$, the singularity is a stable spiral point and the motion is oscillatory. The condition $\kappa^2c^2\geqslant a$ permits calculation of the size of the surge tank to ensure absence of oscillation.

4. THEOREM 2. In the differential equation

$$\frac{d^2y}{dt^2}+2\kappa\frac{dy}{dt}+f(y)=0. \quad \ldots\ldots(1)$$

if (i) $\kappa>0$, (ii) $f(y)=ay+by^3+cy^5+\ldots$, $a, b, c, \ldots, >0$, then for any non-zero initial condition(s) $y=y_0$, $v=v_0$, y is ultimately non-oscillatory provided $\kappa^2>f'(0)$, where $f'(0)$ is the control stiffness* in the equilibrium system.

Proof. From (1)

$$v'=-[f(y)+2\kappa v]/v, \quad \ldots\ldots(2)$$

and in virtue of (ii) the origin $y=v=0$ is the only singular point. By theorem 1, y, and $\dot{y}\to 0$ as $t\to+\infty$. Since y is to be non-oscillatory when $t>t_1$, say, the parameters must be such that the origin is a stable node. Using l'Hospital's rule, we obtain

$$v'(0)=-[f'(0)/v'(0)]-2\kappa, \quad \ldots\ldots(3)$$

so

$$v'(0)^2+2\kappa v'(0)+f'(0)=0,$$

and

$$v'(0)=-\kappa\pm\{\kappa^2-f'(0)\}^{1/2}. \quad \ldots\ldots(4)$$

In virtue of (ii) we have $f'(0)>0$, and since $v'(0)$ must be real, it follows that $\kappa^2\geqslant f'(0)$.

If $|y_0|$ and/or $|v_0|$ is large enough, the v-y curve will pass round the origin a finite number of times before ultimate entry. When $\kappa^2<f'(0)$, the roots of (3) are complex conjugate with negative real parts, the origin is a stable spiral point and the y-t curve is oscillatory but damped. The theorem is valid also if one or more of $b, c, \ldots$, is zero.

By (ii), $f(y)$ has only one real zero, which is at the origin. If the signs of $a, b, c, \ldots$, are optional, $f(y)$ may have any *odd* number of real simple zeros $\alpha_1, \alpha_2, \ldots, \alpha_n$. Each zero corresponds to a singular point of the equation. If α_1 and α_n are either stable spiral points or stable nodes, then $f'(\alpha_1)>0$, $f'(\alpha_n)>0$, and the theorem is valid. For given $a, b, c, \ldots$, the type of singularity depends upon the relative value of κ. The particular singular point about which the motion ultimately subsides depends upon the initial conditions.

If α_1, α_n are cols, the motion is stable for a certain range of initial conditions, but otherwise unstable. N. W. McL.

* In a mechanical system.

A NUMERICAL SOLUTION OF $\frac{d^2y}{dx^2} = F(x)$

By P. Stein

In cases where successive integration of $F(x)$ is not practical, a solution of the equation $\frac{d^2y}{dx^2} = F(x)$ is obtained by approximation. The solution attempted is to find the value of y at selected points in the range of integration. A common procedure is to reduce the problem to the solution of a set of linear simultaneous equations, and these equations are then solved by " relaxation ". It does not appear to be generally known that an explicit and simple solution of these simultaneous equations exists. In this note the detail of such a solution is carried out for functions $F(x)$ as occur in the theory of thin beams.

1. Let $F(x)$ be defined and integrable in the interval $0 \leqslant x \leqslant l$. Let the values of y at $x = 0$ and $x = l$ be y_0 and y_l respectively. Consider the equation

$$\frac{d^2y}{dx^2} = F(x). \quad \text{.....(1)}$$

Let $f'(x) = \int_0^x F(t)\,dt + A$, and $f(x) = \int_0^x f'(t)\,dt + B$. Then $f(x)$ is a solution of (1). If we choose A and B so that $f(0) = y_0$, $f(l) = y_l$, then $y = f(x)$ is the solution required.

Since $f'(x)$ is an integral, it is absolutely continuous. Hence the derivative of $f(x)$ is $f'(x)$ everywhere in the interval. Further, the derivative of $f'(x)$ exists almost everywhere in the interval and is $F(x)$.

If $0 \leqslant a < a + h \leqslant l$ we have

$$f(a+h) - f(a) = \int_0^h f'(a+x)\,dx = -\int_0^h f'(a+h-t)\,dt$$

$$= hf'(a) + \int_0^h tF(a+h-t)\,dt. \quad \text{.....(2)}$$

If $0 \leqslant a - h < a \leqslant l$,

$$f(a-h) - f(a) = -\int_0^h f'(a-h+t)\,dt = -hf'(a) + \int_0^h tF(a-h+t)\,dt. \quad \text{.....(3)}$$

From (2) and (3) we get if $0 \leqslant a - h < a + h \leqslant l$,

$$-f(a-h) + 2f(a) - f(a+h) = \lambda(a)$$

where
$$\lambda(a) = -\int_0^h t\{F(a+h-t) + F(a-h+t)\}\,dt. \quad \text{.....(5)}$$

Suppose we divide the interval $0 \leqslant x \leqslant l$ into $n+1$ equal divisions at points $x = 0, h, 2h, \ldots, nh$. Let y_r be the value of y at $x = rh$. Then $f(rh) = y_r$, $r = 1, 2, \ldots, n+1$, provided $f(0) = y_0$ and $f(l) = y_{n+1} = y_l$. If we now write

$$\lambda_1 = y_0 + \lambda(h); \quad \lambda_r = \lambda(rh), \quad r = 2, \ldots, n-1, \quad \lambda_n = \lambda(nh) + y_l,$$

and we put $a = rh$, $r = 1, 2, \ldots, n$ in succession in (4) we obtain the set of equations

$$\begin{aligned}
2y_1 - y_2 &= \lambda_1, \quad \text{.....(6)} \\
-y_1 + 2y_2 - y_3 &= \lambda_2, \\
\ldots \quad \ldots \quad \ldots &\quad \ldots \\
-y_{r-1} + 2y_r - y_{r+1} &= \lambda_r, \\
\ldots \quad \ldots \quad \ldots &\quad \ldots \\
-y_{n-1} + 2y_n &= \lambda_n.
\end{aligned}$$

If D_n is the determinant of order n given by

$$D_n = \begin{vmatrix} 2, & -1, & 0, & \dots \\ -1, & 2, & -1, & \dots \\ 0, & -1, & 2, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix},$$

and if A_r is the determinant obtained from D_n by replacing its r-th column by $\lambda_1, \lambda_2, \dots, \lambda_n$, the solution of (6) is given by

$$y_r = A_r/D_r, \quad r = 1, 2, \dots, n. \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(7)$$

It is readily seen that $D_n = 2D_{n-1} - D_{n-2}$. Since $D_1 = 2$, $D_2 = 3$, we have immediately $D_n = n+1$. Also if $A_{i,j}$ is the cofactor of the element in the i-th row and the j-th column of D_n, we have that $A_{i,j} = A_{j,i}$ and if $i \leqslant j$, $A_{i,j} = D_{i-1} D_{n-j}$. Combining these results with (7) we have

$$y_r = (n+1)^{-1}\{(n-r+1)(\lambda_1 + 2\lambda_2 + \dots + r\lambda_r) + r((n-r)\lambda_{r+1} + (n-r-1)\lambda_{r+2} + \dots \lambda_n)\}. \qquad \dots\dots(8)$$

2. To complete the solution we have to find a suitable expression for $\lambda_r = \lambda(rh)$ as given by (5).

Suppose that $F(x)$ can be expanded in a Taylor series at each point $x = rh$, $r = 1, 2, \dots, n$ which is convergent in the range $(r-1)h \leqslant x \leqslant (r+1)h$. Then integrating by parts we have

$$\lambda_r = -2\left\{\frac{h^2}{2} F(rh) + \frac{h^4}{4} F''(rh) + \frac{h^6}{6} F^{(\mathrm{IV})}(rh) + \dots\right\}. \qquad \dots\dots\dots\dots(9)$$

A case of interest that occurs in the theory of thin elastic beams is when $F(x)$ and its derivatives are " piece-wise smooth ". That is to say, for each x in $0 \leqslant x \leqslant l$, $F^{(n)}(x-)$ and $F^{(n)}(x+)$ exist and $F^{(n)}(x+) = F^{(n)}(x-)$ except at a finite number of values of x, at which points $|F^{(n)}(x+) - F^{(n)}(x-)|$ is finite. We proceed to find the value of λ_r in this case.

We require the following lemma.

Lemma. If $\phi(x)$ is a function which is piece-wise smooth in $a-h \leqslant x \leqslant a+h$, if at a point of discontinuity we define $\phi(x)$ as $\phi(x-)$, if the salti occur at points ξ_k, η_k where $a-h \leqslant \xi_1 < \xi_2 < \dots < \xi_p < a \leqslant \eta_1 < \eta_2 < \dots < \eta_r < a+h$, if

$$A_n = \sum_{k=1}^{p} (\xi_k - a + h)^n \{\phi(\xi_k+) - \phi(\xi_k)\}$$

and

$$B_n = \sum_{k=1}^{q} (a + h - \eta_k)^n \{\phi(\eta_k+) - \phi(\eta_k)\},$$

then

$$\int_0^h \frac{t^{n-1}}{(n-1)!}\phi(a+h-t)\,dt = \frac{h_n}{n!}\phi(a) + \frac{B^n}{n!} + \int_0^h \frac{t^n}{n!}\phi'(a+h-t)\,dt,$$

and

$$\int_0^h \frac{t^{n-1}}{(n-1)!}\phi(a-h+t)\,dt = \frac{h^n}{n!}\phi(a) - \frac{A_n}{n!} - \int_0^h \frac{t^n}{n!}\phi'(a-h+t)\,dt.$$

Both these results follow readily on integration by parts.

Returning to (5) we have from this lemma, with $F(x)$ in place of $\phi(x)$,

$$-\lambda(a) = 2\frac{h^2}{2!}F(a) + \frac{B_2}{2!} - \frac{A_2}{2!} + \int_0^h \frac{t^2}{2!}\{F'(a+h-t) - F'(a-h+t)\}\,dt.$$

On applying this lemma over again to the functions $F'(a+h-t)$ and $F'(a-h+t)$, we have

$$-\lambda(a) = h^2 F(a) + \frac{B_2 - A_2}{2!} + \frac{B_3 - A_3}{3!} + \int_0^h \frac{t^3}{3!}\{F''(a+h-t) + F''(a-h+t)\}dt. \quad (10)$$

Applying (10) in succession to F'' and F^{iv}, etc., we arrive at the result

$$-\lambda(a)=2\sum_{s=1}^{N}\frac{h^{2s}}{(2s)!}F^{2s-2}(a)+\int_0^h\frac{t^{2N+1}}{(2N+1)!}\{F^{(2N)}(a+h-t)+F^{(2N)}(a-h+t\}\,dt$$
$$+\sum_{s=2}^{2N+1}\frac{B_s}{s!}+\sum_{s=2}^{2N+1}(-)^{s-1}\frac{A_s}{s!}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

If $a=rh$, this gives λ_r.

3. In practice $\lambda(rh)$ is generally taken as $-h^2F(rh)$. It is desirable to form an estimate of the resulting error in the values of y_r.

Suppose F and F' are continuous. Let $e(a)$ be the error in (a). From (10),

$$e(a)=\int_0^h\frac{t^3}{3!}\{F''(a+h-t)+F''(a-h+t)\}\,dt.$$

If M is the maximum modulus of $F''(x)$ in $0<x<l$, then

$$e(a)\leqslant 2M\int_0^h\frac{t^3}{3!}\,dt=Mh^4/12. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

Now $(n-r+1)(1+2+3+\ldots+n)+r\{(n-r)+(n-r-1)+\ldots+1\}$
$$=\tfrac{1}{2}r(n+1)(n-r+1). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13)$$

Since the equations in y_r are linear, from (8), (12) and (13) we have that the maximum error in y_r is less than $r(n-r+1)Mh^4/24$. Since $h=l/(n+1)$ and $r(n-r+1)<\frac{1}{2}(n+1)^2$, this gives the maximum error in y_r as less than $Ml^4/48(n+1)^2$.

In the case where $F(x)$ and $F'(x)$ have discontinuities, we have to add to this item the effect of neglecting the salti. Suppose we have a saltus L at $x=\xi$, $sh\leqslant\xi<(s+1)h$. This saltus gives rise to a term in A_2 in λ_s, and a term in B_2 in λ_{s+1}. Suppose $s<r$. Let $\xi=sh+th$, $0\leqslant t<1$. Then from (8) the resulting error in y_r corresponding to this saltus is

$$\left|\frac{n-r+1}{2(n+1)}\{st^2-(s+1)(1-t^2)\}Lh^2\right|$$
$$\leqslant\frac{n-r+1}{2(n+1)}\{s\mid t^2-(1-t^2)\mid+1\}h^2\mid L\mid$$
$$\leqslant\frac{n-r+1}{2(n+1)}(s+1)h^2\mid L\mid\leqslant\frac{n-r+1}{2(n+1)}rh^2\mid L\mid\leqslant\tfrac{1}{4}(n+1)h^2\mid L\mid.$$

If now m is the number of salti of F, and M' their maximum modulus, then the resulting error of neglecting the salti in λ_r is of order $mM'l^2/4(n+1)$. This is of a larger order in $1/n$ than the order of the previous error and may be too large to be neglected.

A similar argument will show, using the inequality

$$(1-t)^3+t^3\leqslant 1, \quad 0\leqslant t<1,$$

that the maximum error in neglecting the terms in the salti of F'' introduces an error less than $m'M''l^3/2(n+1)^2$, where m' is the number of discontinuities in F', and M'' is the maximum modulus of the salti of F'.

4. We have assumed that the end conditions were that y was given at $x=0$ and $x=l$. In the theory of thin elastic beams, this is the case of a beam supported at two points. However, the method may be adopted to various end conditions. Thus in the case of a beam clamped at $x=0$ and free at $x=l$, we may carry out a preliminary calculation as follows. We extend the length of integration from $x=0$ to $x=-h$. We suppose that from $x=0$ to $x=-h$, $F(x)=0$. In that case for $-h\leqslant x<0$, $y'(x)=y_0$ and so $y(-h)=-hy_0'+y_0$.

We now write down the formula for y_0 (with obvious modifications), using $n+2$ points, with $x=-h$ and $x=l$ as end points. Since y_0 is known, the equation we get may be used to calculate $y_{n+1}=y_l$. With y_l known the formula (8) may now be used.

In the case of a beam clamped at $x=0$ and supported at $x=l$,

$$F(x)=F^*(x)+ax+b,$$

where $F^*(x)$ may be computed from the load distribution and the terms $ax+b$ depend on the reactions at the supports. One relation between a and b can be found from a knowledge of the bending moment at $x=l$. Another relation may be obtained by extending the range of integration to $x=-h$, as above.

The case of a beam clamped at both ends may be similarly treated. $F(x)=F^*(x)+ax+b$. We extend the range of integration at one end to $x=-h$ and at the other end to $x=l+h$. We then write down equations for y_0 and y_l, and these equations enable us to determine a and b.

Finally it may be pointed out that (8) need not be used to calculate all the values of y_r. The equations (6) lend themselves to a "stepping off" process. We calculate y_1 from (8). y_2 may then be derived from the first of equations (6). With y_1 and y_2 known, y_3 may be determined from the second of the equations (6), and so on.

P.S.

1828. The flashpoint of anger to most industrial workers is the £9 17s. 8d. "average wage packet". They say: "Where do you get that rubbish from?" It comes from the Minister of Labour, and is the average wage of all industrial workers. It means that many earn well above £9 17s. 8d. a week. And many below that figure. The two together is the average pay packet. It works this way. Hutton, England's captain, scores 50, then 60, and 100 in three innings. Thus his average score is 70. And all Yorkshire would write in if we said it was less.

1829. During the tests Colonel Stapp's body underwent a pressure of 22 times the force of gravity and assumed the weight of 3,960 lb.—*The Times*. [Per Mr. J. H. Pearce.]

1830. My sister would, I know, join in the protest of "Danish Editor" against British weights and measures. She was taught by a Swiss governess:

16 ozzies = 1 label.
14 labels = 1 stone.
28 labels = 1 quarter.
112 labels = 1 quart.

She concluded that mathematics were unfathomable.—Isla Macpherson, in *The Observer*, June 27, 1954. [Per Mrs. W. A. Fish.]

1831. Where the frequency of the service is 30 minutes or more conductors will exercise discretion to vary standing passengers at all times . . . in order to avoid hardship.—L. T. E. Notice in country buses. [Per Mr. J. C. W. De la Bere.]

1832. Government Geologist George W. Swindel . . . examined the black stone and pronounced it "a smooth angular rhombus† with some of its corners broken off".—*Time*, December 13, 1954. [Per Professor H. S. M. Coxeter.]

† A six sided solid whose faces are parallelograms.

SCHUR'S INEQUALITY

By G. N. Watson.

In the *Gazette* for December 1953, I gave an account of all that I knew about Schur's inequality, namely

If $\mu \geqslant 0$ and x, y, z are all positive, then

$$f(x, y, z;\ \mu) \equiv x^\mu(x-y)(x-z) + y^\mu(y-z)(y-x) + z^\mu(z-x)(z-y) \geqslant 0,$$

with equality occurring when and only when $x = y = z$.

In my previous paper, after giving an unsymmetrical proof of the inequality, I quoted the result (apparently due to S. Barnard and J. M. Child) that

$$f(x, y, z;\ \mu) \equiv xyz f(1/x, 1/y, 1/z;\ -\mu - 1),$$

whence it follows that the inequality holds for all negative values of $\mu + 1$. I then gave a (new) *symmetrical* proof that the inequality also holds for the range $-1 \leqslant \mu \leqslant 0$, so that the inequality has now been shewn to hold for *all real values* of μ.

The situation, however, is not satisfactory because of the lack of symmetry in the proof for positive values of μ (and consequently also for negative values of $\mu + 1$). The only contribution which I have been able to make hitherto towards filling these gaps was to give two identities which yielded symmetrical proofs for the special cases $\mu = 1$ and $\mu = 2$; and, of course, by the theorem on inversion, these identities immediately yield symmetrical proofs for the special cases $\mu = -2$ and $\mu = -3$.

I have recently noticed that a simpler form is obtainable for the identity associated with the case $\mu = 1$; and this simpler form has suggested a symmetrical proof of Schur's inequality for the range $0 \leqslant \mu \leqslant 1$. Consequently (with the help of the theorem on inversion) we can prove Schur's theorem symmetrically for any value of μ in the range $-2 \leqslant \mu \leqslant 1$. For values of μ outside this range (2 and -3 excepted), I have not succeeded in constructing a symmetrical proof.

Accordingly I now give in full the analysis by which the original identity associated with $\mu = 1$ was constructed (my previous paper contained only an epitome of the work). I then derive the simpler form of the identity by a device which is not without intrinsic interest as an example in the use of complex numbers. Finally I explain how the simpler identity can be modified so as to cope with values of μ in the range $0 \leqslant \mu \leqslant 1$.

When we replace x by $t(y+z)$ in $f(x, y, z;\ 1)$, we obtain an expression of degree 3 in y and z; this expression has $y+z$ as a factor. We rearrange the complementary (quadratic) factor as the square of an expression which is linear in y and z added to a term in yz. This procedure gives

$$f(x, y, z;\ 1) \equiv (y+z)\{(t+1)(t-1)^2(y-z)^2 + (2t-1)^2 tyz\};$$

we now substitute $x/(y+z)$ for t and clear of fractions, so that we have

$$(y+z)^2 f(x, y, z;\ 1) \equiv (x+y+z)(y-z)^2(x-y-z)^2 + xyz(2x-y-z)^2.$$

The identity given in my previous paper is obtained by changing x, y, z cyclically and summing.

The factor $y+z$ can evidently be removed from the identity which has just been written down. We endeavour (successfully) to effect the removal in such a way as to exhibit the resulting expression on the right in a form which is ostentatiously non-negative.

The expression on the right of the identity can be written in the form of the sum of

$$(y+z)(y-z)^2(x-y-z)^2$$

and

$$x(y-z)^2(x-y-z)^2+xyz(2x-y-z)^2.$$

The former of these expressions gives no trouble. In the latter (after discarding the factor x) we replace x, y, z by ξ^2, η^2, ζ^2, so that we are faced with the problem of removing the factor $\eta^2+\zeta^2$ from the expression

$$(\eta^2-\zeta^2)^2(\xi^2-\eta^2-\zeta^2)^2+\eta^2\zeta^2(2\xi^2-\eta^2-\zeta^2)^2.$$

This expression is the product of the pair of conjugate factors

$$(\eta^2-\zeta^2)(\xi^2-\eta^2-\zeta^2)\pm i\eta\zeta(2\xi^2-\eta^2-\zeta^2),$$

one of which must be divisible by $\eta+i\zeta$, while the conjugate factor must be divisible by $\eta-i\zeta$. There is no difficulty in verifying that

$$(\eta^2-\zeta^2)(\xi^2-\eta^2-\zeta^2)+i\eta\zeta(2\xi^2-\eta^2-\zeta^2)\equiv(\eta+i\zeta)\{\eta(\xi^2-\eta^2)+i\zeta(\xi^2-\zeta^2)\}.$$

We take the identity conjugate to this and multiply ; when we revert to the original notation, we see that we have proved that

$$(y-z)^2(x-y-z)^2+yz(2x-y-z)^2\equiv(y+z)\{y(x-y)^2+z(x-z)^2\}.$$

We now collect our results, and we find that

$$(y+z)f(x,y,z\,;\ 1)\equiv xy(x-y)^2+xz(x-z)^2+(y-z)^2(x-y-z)^2.$$

By changing x, y, z cyclically and summing, we deduce my new identity, namely

$$2(\Sigma x)f(x,y,z\,;\ 1)\equiv 2\,\Sigma yz(y-z)^2+\Sigma(y-z)^2(x-y-z)^2;$$

and this identity immediately supplies a symmetrical proof of Schur's inequality in the special case $\mu=1$.

Our next step is empirical ; with arbitrary X, Y, Z, we take the expression

$$2(\Sigma x)\{X(x-y)(x-z)+Y(y-z)(y-x)+Z(z-x)(z-y)\}$$

and subtract the expression

$$\Sigma(yZ+zY)(y-z)^2+\Sigma(y-z)(Y-Z)(x-y-z)^2$$

from it. The result reduces to

$$\Sigma x(y-z)(yZ-zY).$$

We now write x^μ, y^μ, z^μ in place of X, Y, Z throughout, and we immediately obtain the identity

$$2(\Sigma x)f(x,y,z\,;\ \mu)$$
$$\equiv\Sigma(yz^\mu+zy^\mu)(y-z)^2+\Sigma(y-z)(y^\mu-z^\mu)(x-y-z)^2+\Sigma x(y-z)(yz^\mu-zy^\mu).$$

Each of the three terms in the first sum on the right is always non-negative; each of the three terms in the second sum is always non-negative provided that $\mu\geqslant 0$; and each of the three terms in the third sum is always non-negative provided that $\mu\leqslant 1$. Moreover, for all of the nine terms in the three sums to vanish, it is necessary and sufficient for the three positive numbers x, y, z to be all equal. The identity consequently supplies a symmetrical proof of Schur's inequality for all values of μ which lie in the range $0\leqslant\mu\leqslant 1$. This is the result which I set out to prove.

G. N. W.

MATHEMATICAL NOTES

2530. *A pentagonal tessellation.*

My colleague Mr. R. C. Lyness noticed this pattern on the floor of a school in Germany. It has also appeared in a crossword puzzle in *The Listener*. It might be used in " patchwork " as a change from the traditional hexagon.

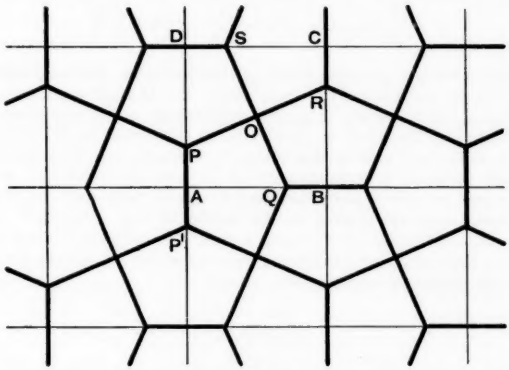


FIG.

It can be constructed easily on squared paper. If $ABCD$ is a typical square, points P, Q, R, S on DA, AB, BC, CD such that $PA = QB = RC = SD$ are marked ; PR, QS of course are at right angles and pass through the centre O of the square.

A class might discuss the particular cases arising as P moves from A to D : how the areas of the pentagons and hexagons remain constant and equal to a^2, $2a^2$ respectively, where $AB = a$. The lengths of the sides of the pentagon might be expressed in terms of a and p $(= AP)$, or a and θ $(\angle OQA)$ and the constancy of area verified. The perimeter of the hexagon might be found, and the condition for its value to be a minimum ; this occurs when the triangle PQP' is equilateral, and the squares $PQRS$ then form with the triangles PQP' a mixed regular tessellation.

Another simple problem is to find the position of P which makes the pentagon equilateral.

A. P. R.

2531. *A note on numerical integration.*

I should like to draw the attention of readers of the *Gazette* to the utility of the simplest case of Tchebychef's formulae of approximate integration, particularly in relation to its use conjointly with Simpson's well-known formula. Statements of Tchebychef's formulae are to be found in most text-books on numerical calculus (see, for example, *Numerical Mathematical Analysis*, by J. B. Scarborough, p. 155) but any possible usefulness is seldom indicated. To the writer's knowledge the present application has not been published previously.

Simpson's rule may be, essentially, expressed as

$$\int_0^2 f(x)\,dx = \tfrac{1}{3}[f(0) + 4f(1) + f(2)],$$

and it is easily shown that the error is of the order $\Delta^4 f(0)/90$, where Δ is the usual advancing difference operator over an interval of unity. The high accuracy of Simpson's rule, particularly in regard to the fact that only *three* values of $f(x)$ are involved, has often been commented on (see, for example, *Numerical Calculus*, by W. E. Milne, p. 121).

The first Tchebychef rule may be expressed, taking 0 to 2 as the range of integration, as

$$\int_0^2 f(x)\,dx = f\left(1-\frac{\sqrt{3}}{3}\right)+f\left(1+\frac{\sqrt{3}}{3}\right).$$

A little algebra shows that the error in this formula, which involves only *two* values of $f(x)$, is, surprisingly, numerically less than that of the comparable Simpson rule; and, what is more important, is of opposite sign. The numerical value of the error is $\frac{1}{135}\Delta^4 f(0)$, to this order of differences.

Thus, not only does this formula supply a useful check on the value of the integral, but further, a weighted mean in the proportion 3 to 2 between results obtained by the Tchebychef and Simpson rules may be expected to be an even better approximation to the value of the integral.* Incidentally, when the range of integration is not a simple multiple of some fundamental unit there is little to choose between the simplicity of either of these rules.

As a simple illustration consider

$$\int_0^{\pi/2} \sin\theta\,d\theta.$$

Simpson's rule, using *three* values of the integrand over the entire range gives 1·00228, the Tchebychef rule, using only *two* values, gives 0·99847. The weighted mean is 1·00000, with an error of less than a unit in the last decimal place.

Acknowledgement is made to the Chief Scientist, Ministry of Supply, for permission to publish this note. S. J. TUPPER.

2532. *A note on multiplication.*

The following method of multiplication is suggested as being quicker and less tedious than the standard one in use. It is merely an application of the method of picking out coefficients used when multiplying polynomials in elementary algebra, but it enables the product of quite large numbers to be obtained without any written working. An example will illustrate the procedure.

$$\begin{array}{r} 1{,}234 \\ 5{,}678 \\ \hline 7{,}006{,}652 \end{array}$$

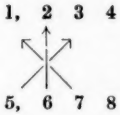
$1{,}234 \times 5{,}678$ may be considered as

$$(1 \text{ thousand} + 2 \text{ hundreds} + 3 \text{ tens} + 4 \text{ units}) \times$$
$$(5 \text{ thousands} + 6 \text{ hundreds} + 7 \text{ tens} + 8 \text{ units}).$$

The units of the product originate from the multiplication of 4 by 8, giving 32 units. We write 2 and carry 3 tens. Tens arise from the 3 carried together with $8\times 3 + 7\times 4$, making 55 in all. We write 5 and carry 5 hundreds. Then for the hundreds we have 5, $+8\times 2 = 21$, $+7\times 3 = 42$, $+6\times 4 = 66$. Write 6

* In fact the weighted mean indicated provides an integration formula in error by a term of $\Delta^6 f(0)$ only.

carry 6, etc. In practice the operations are carried out mentally quite easily. As an illustration of the symmetry of the cross-multiplications involved the arrows show the manner in which the tens of thousands (excluding those previously carried) are derived in the calculation.



After the working of a few examples the above result would be obtained within about 40 seconds.

The method is applicable to calculations of a considerable size, for example,

$$\begin{array}{r} 1{,}234{,}512{,}345 \\ 2{,}345{,}123{,}451 \\ \hline 2{,}895{,}083{,}850{,}808{,}502{,}595 \end{array}$$

However, if preferred, for large numbers or those containing several 8's and 9's a slight modification of the working may be employed. By this we would write

$$\begin{array}{r} 21{,}326 \\ 4{,}174 \\ \hline 8,\ 6,\ 27,\ 26,\ 51,\ 32,\ 50,\ 24 = 89{,}014{,}724 \end{array}$$

The lower numbers are obtained by cross-multiplication and addition as before, but no carrying is performed between operations.

Thus, $4 \times 6 = 24$ units,
$4 \times 2 + 7 \times 6 = 50$ tens, etc.

These numbers are then compounded to form the final result.

L. MARDER.

2533. *A property of the quadrilateral.*

Let $A_1A_2A_3A_4$ be a quadrilateral and let $B_1A_1A_2$, $B_2A_3A_2$, $B_3A_3A_4$, $B_4A_1A_4$ be similar triangles, right-angled at the B, and constructed to lie outside the quadrilateral. We shall prove that the diagonals B_1B_3, B_2B_4 of the quadrilateral $B_1B_2B_3B_4$ are equal and that the angles between them are twice the acute angles of any one of the right-angled triangles.

Inglada (*Math. Gaz.* 36 (1952), 55–6) refers to the particular case of this result in which the right-angled triangles are isosceles as van Aubel's theorem. Neumann (*Journ. London Math. Soc.* 16 (1941), 230–45) gave an interesting account of the generalisation of the latter theorem to n-gons, in which the triangles are still isosceles though not always right-angled (see also Baker, *ibid.* 17 (1942), 162–4, and Neumann, *ibid.* 17 (1942), 165–6). My own generalisation is not included in Neumann's.

Let z_1, z_2, z_3, z_4 be the complex numbers representing A_1, A_2, A_3, A_4, which we may suppose to go clockwise round the quadrilateral. Let α, β be the angles $A_2A_1B_1$, $A_1A_2B_1$ respectively. The mid-point of A_1A_2 is $\frac{1}{2}(z_1+z_2)$ and so B_1 is

$$\tfrac{1}{2}(z_1+z_2)+\tfrac{1}{2}(z_2-z_1)e^{2\alpha i}=\tfrac{1}{2}z_1(1-e^{2\alpha i})+\tfrac{1}{2}z_2(1+e^{2\alpha i}).$$

Similarly B_2 is $\frac{1}{2}z_2(1-e^{2\beta i})+\frac{1}{2}z_3(1+e^{2\beta i})$, B_3 is $\frac{1}{2}z_3(1-e^{2\alpha i})+\frac{1}{2}z_4(1+e^{2\alpha i})$ and B_4 is $\frac{1}{2}z_4(1-e^{2\beta i})+\frac{1}{2}z_1(1+e^{2\beta i})$.

From these we have

$$\overrightarrow{B_1B_3}=\tfrac{1}{2}(z_3-z_1)(1-e^{2\alpha i})+\tfrac{1}{2}(z_4-z_2)(1+e^{2\alpha i})$$

and

$$\overrightarrow{B_2B_4}=\tfrac{1}{2}(z_1-z_2)(1+e^{2\beta i})+\tfrac{1}{2}(z_4-z_2)(1-e^{2\beta i}).$$

Now $e^{2(\alpha+\beta)i}=e^{\pi i}=-1$ and so $e^{2\alpha i}\overrightarrow{B_2B_4}=\overrightarrow{B_1B_3}$, and the result follows.

If the triangle $B_1A_1A_2$ is constructed on the opposite side of A_1A_2, *i.e.* " inside " $A_1A_2A_3A_4$, and similarly for the other triangles, the result is still true.

I am indebted to Dr. J. W. Robertson for drawing my attention to van Aubel's theorem and to Dr. L. S. Goddard for the reference to Neumann and Baker above. E. M. WRIGHT.

2534. *A geometrical picture of the set of all real quadratic polynomials.*

I describe here a geometrical representation of some familiar properties of, and relations between, real quadratic polynomials. In what follows I shall mean by homogeneous coordinates (x, y, z) those obtained from ordinary rectangular cartesians (X, Y) by writing $X=x/z$, $Y=y/z$.

The point with homogeneous coordinates (a, b, c) is to be called the *representative point* of the polynomial ax^2+bx+c. If $c=0$ (when the polynomial has a zero root), it is a point at infinity. Proportional polynomials are represented by the same point. The parabola with equation $y^2=4xz$ is to be called the *discriminating parabola*.

I. *The representative points of polynomials with real roots lie outside or on the discriminating parabola, and those of polynomials with imaginary roots lie inside. A polynomial has a repeated root if and only if its representative point lies on the discriminating parabola.*

This is a restatement of the result that the polynomial ax^2+bx+c has real distinct, coincident, or imaginary roots according as b^2 is greater than, equal to, or less than $4ac$.

II. *Two polynomials have a common real root if and only if their representative points lie on a tangent to the discriminating parabola.*

Suppose the polynomials ax^2+bx+c, $a'x^2+b'x+c'$ have the real root α in common. Then $a\alpha^2+b\alpha+c=0$, $a'\alpha^2+b'\alpha+c'=0$, so that (a, b, c) and (a', b', c') lie on the line $\alpha^2x+\alpha y+z=0$, which is a tangent to the parabola $y^2=4xz$. Conversely every tangent to the discriminating parabola has an equation of the form $\alpha^2x+\alpha y+z=0$, and if the representative points of two polynomials lie on this line the polynomials have the root α in common. Alternatively we may observe that two polynomials ax^2+bx+c, $a'x^2+b'x+c'$ have a common root if and only if their ***resultant*** vanishes ; *i.e.* if and only if

$$(ac'-a'c)^2=(bc'-b'c)(ab'-a'b).$$

Suppose ax^2+bx+c has real roots, so that (a, b, c) lies outside the discriminating parabola ; then the representative points of all polynomials which have a root in common with ax^2+bx+c lie on the line-pair given by

$$(az-xc)^2=(bz-yc)(ay-xb),$$

and this is the equation of the pair of tangents from (a, b, c) to $y^2=4xz$.

We note that the points of intersection A, B (say) with the y-axis of the pair of tangents from (a, b, c) represent the two linear polynomials whose roots are the two roots of ax^2+bx+c. It follows at once that these roots $(\alpha, \beta$ say) are represented by minus the reciprocals of the y-coordinates of A and B.

III. *Two polynomials are apolar if and only if their representative points are conjugate points with respect to the discriminating parabola.*

This follows at once since the polynomials ax^2+bx+c, $a'x^2+b'x+c'$ are apolar if and only if $2(ac'+a'c)=bb'$.

A polynomial is of course self-apolar if and only if it has a repeated root.

Geometrically the following fact is obvious, namely that all polynomials which are apolar to a polynomial with imaginary roots have real roots. Therefore in particular the unique polynomial which is apolar to two given polynomials (their *jacobian*) has real roots provided one of the given polynomials has imaginary roots.*

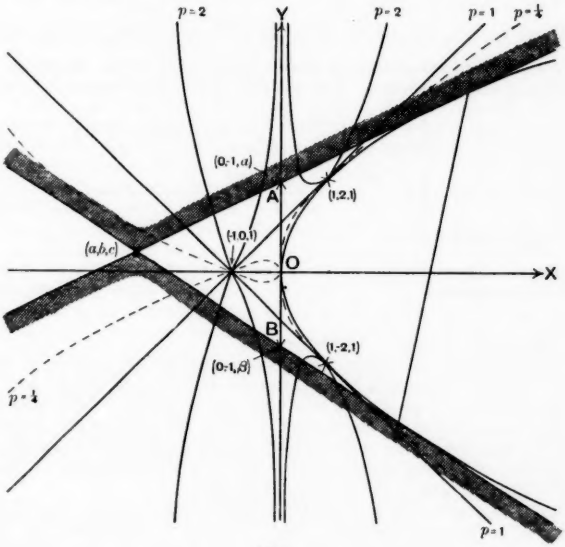


FIG.

The derivative of ax^2+bx+c is represented by the point of intersection with the y-axis of the polar of (a, b, c).

IV. *The representative points of polynomials whose (real) roots separate the (real) roots of a given polynomial ax^2+bx+c fill that region enclosed between the tangents from (a, b, c) to the discriminating parabola which does not contain* 0. (This region is shaded on the figure.)

* This result is often used. (Of course, it is simple to prove algebraically, but the geometry makes it obvious.) *E.g.* a standard method of transforming the integral $\int(ax^2+bx+c)^{-1}(a'x^2+b'x+c')^{-\frac{1}{2}}\,dx$, where $b^2<4ac$, is to put $x=(\lambda+\mu t)/(1+t)$, where λ, μ are the roots of the jacobian of ax^2+bx+c and $a'x^2+b'x+c'$.

Now the roots of $a'x^2+b'x+c'$ separate the roots of ax^2+bx+c if and only if the resultant of the two polynomials is negative, *i.e.* if and only if

$$(ac'-a'c)^2-(bc'-b'c)(ab'-a'b)<0.$$

But this means that the product of the perpendicular distances of (a', b', c') from the pair of tangents $(az-xc)^2=(bz-yc)(ay-xb)$ is negative; whence the required result follows.

It is interesting to locate the representative points of polynomials whose roots are integral powers of the roots of a given polynomial ax^2+bx+c. (We shall assume here that the roots of ax^2+bx+c are real. A discussion could be supplied for the case of conjugate imaginary roots.)

Suppose firstly that ax^2+bx+c has two positive roots α, β. The coordinates of the representative point of a polynomial whose roots are some power of α, β satisfy the relations $-y/x=\alpha^r+\beta^r$, $x/z=1/\alpha^r\beta^r$. Whence, eliminating r, we have $y/z=-\left[\left(\frac{x}{z}\right)^p+\left(\frac{x}{z}\right)^{1-p}\right]$, $p=\log\alpha/\log\alpha\beta$.

Without going into details we state the following more general result:

V. *The representative points of those polynomials whose roots are integral powers of the (real) numbers α, β lie on the curve whose branches are given by*

$$y/z=\pm\left[\left(\frac{x}{z}\right)^p+\left(\frac{x}{z}\right)^{1-p}\right],\quad \left(\textit{defined for } \frac{x}{z}>0\right),$$

$$y/z=\pm\left[\left(-\frac{x}{z}\right)^p-\left(-\frac{x}{z}\right)^{1-p}\right],\quad \left(\textit{defined for } \frac{x}{z}<0\right),$$

where $p=\log|\alpha|/\log|\alpha\beta|$.

(The curves corresponding to $p=1$ (or 0), $p=2$ (or -1), $p=\frac{1}{2}$ (or $\frac{3}{2}$) are sketched in the figure.) HAZEL PERFECT.

2535. *Mr. C. V. Durell's "Genarith".*

This prolific writer of mystery books—it would be wrong to call them "thrillers"—fully maintains his reputation in the work before us,* and the reader who delights in problems will find here a plenty. Who are the characters? how many of them are there? what is the plot of the story?—these are some of the questions which the mysteriologist will approach with a simple interest which will intensify in proportion to his success in discounting the false clues with which the volume abounds.

Mr. Durell has adopted a somewhat statistical and episodic style which does not make for easy reading; each incident is classified and numbered and presented in a completely objective and baldly matter-of-fact form in the minimum of words. The author's descriptive style is frankly lamentable and one feels that he has not succeeded in making his characters live. He veils them in an anonymity which cuts across the reader's sympathy, referring to them impersonally as "a man" or as "*A*, *B* and *C*"; thus it is not easy always to relate a character to the incidents of his career, more especially since the author gives no clue to the thought-processes or motives leading up to the events he narrates. He has, too, a disconcerting habit of suddenly shifting the locale of his story without the slightest warning. Thus on p. 141 an explorer investigating the sources of the Amazon appears to have been transported thither from the Ganges in the twinkling of an eye, and on p. 300

* *General Arithmetic for Schools*, by Clement V. Durell, pp. xvi, 411. 21st impression, 1951 (Bell).

he gets himself unaccountably mixed up with Feddens and qirats in Egypt. There is similarly the Home Office inspector who travels vaguely round the country enquiring into the Local Government of unsuspecting towns, with especial reference to their finances—p. 142 Winchester, p. 268 Oxford, p. 386 Leeds, p. 388 Poole. These people certainly get about!

The main action of the book takes place in this country and in France, as we may deduce from the frequency with which the characters are called on to exchange English money into French and *vice versa*: the period was evidently before the war, since cigarettes were 20 for 1/- (p. 27). As might be expected, the most clearly defined personality is the villain. He begins in a small way as a barrow-boy, putting only 12 peaches in boxes ostensibly containing 18 (p. 30) and selling oranges two-ninths of which are bad (p. 42); the absence of any mention of nylons confirms our estimate of the period. Considering the company he keeps—in one chapter the words "common", "vulgar" and "improper" recur monotonously—how can he be expected to remain "integer vitae, scelerisque purus"? From his first petty defalcations he is able to make enough to buy a small milk-round; he waters the milk and narrowly escapes well-deserved ruin when he spills some of it (p. 175). Even when he plays cricket he cannot refrain from falsifying his average (p. 236), and he advertises a car for sale claiming for it a performance far in excess of the truth (p. 116). For a time his career is obscure. There is a cryptic reference to a Derby run in record time (p. 237) on which he presumably wins a packet, since he gives a celebration dinner but fails to pay nearly half the bill (p. 256) though he spends £96 on shirts (p. 116). In an access of remorse he pays £1195 as Conscience Money to the Exchequer (p. 258), but he ends miserably with an overwhelming number of bankruptcy orders against him (p. 142).

A less clearly drawn character is the man who is Something in the City. His father encouraged him to save in his boyhood (p. 157), and later in life he was able to do a little money-lending at 12½% (p. 152). He has a setback when a Bank holding some of his savings fails (p. 158), and after dabbling in insurance (p. 165) and bill discounting (p. 221) he loses money in two speculations obviously suggested by the villain (p. 257). But he is not dismayed: some lucky deals in foreign stocks (p. 366) and a profitable foreign exchange operation (p. 398) set him on his feet. Only once does he fall foul of the law, when he is trapped by a policeman for travelling at 68 m.p.h. in a built-up area (p. 378).

We can do no more than mention the scientist with his germ-cultures (p. 147), the swimming enthusiast who is always building swimming baths and emptying and refilling them (*passim*) or the College Youths whose sense of timing is poor (p. 13); but attention must be drawn to the book's most striking feature—the complete absence of love-interest. On p. 50 there is certainly mention of a co-educational school where doubtless began a boy-and-girl romance which culminated in the marriage (on p. 51) of a man of 27 to a woman of 24, but there is not a word about the courtship nor of the outcome of the marriage except that it lasted 54 years and a hint that the wife joined a committee (p. 265): perhaps it was that Road Safety Committee which discovered that if you don't want to be killed on the roads you should not be a child under ten (p. 102).

A remarkable book: and if you think this review is sketchy see if you can do any better yourself!

B. A. SWINDEN.

2536. *A note on Latin squares.*

A magic square is an arrangement of n^2 different integers, usually consecutive, in a square such that the sums of each row, each column and each

of the leading diagonals are the same. Magic squares have also been constructed with similar properties in relation to subtraction, multiplication and division.

Magic squares were known in ancient times in China and India and, engraved on metal or stone, are still worn as amulets. The first known writer in Europe was Emanuel Moschopoulos, a Greek of Constantinople (*c.* A.D. 1300). C. Agrippa (1486–1535) constructed magic squares of orders 3–9 and associated them with the seven astrological planets Saturn, Jupiter, Mars, Sun, Venus, Mercury, Moon. A magic square of order 4 is shown in A. Dürer's picture " Melancholy ", and whether by design or accident the date of the picture—1514—is given by the four middle digits of the bottom row.

Methods of construction were given by de la Loubère in the seventeenth century, and since then the construction and classification of magic squares, including " nasik " squares in which the constant sum property is possessed also by the broken diagonals, has been a favourite mathematical pastime. Accounts appear in W. W. R. Ball's *Mathematical Recreations* and in a book of the same title by M. Kraitchik. Notes on magic squares are still frequently published. For example, *Scripta Mathematica* has contained a number of articles since 1940, and the same applies to the *Mathematical Gazette*.

The allied problem of the Latin square seems first to have attracted the attention of Euler in an attempt to solve the problem of the 36 officers. It is required to arrange 36 officers of 6 different grades and drawn from 6 different regiments in such a way that in each row and in each column there occur six officers of different grades and drawn from the 6 different regiments. In the generalised problem of n^2 officers, a square is formed so that each of the first n Latin letters appears just once in each row and in each column. The solution of the problem depends upon finding an " orthogonal " square such that when the two squares are superposed there is one and only one conjunction of every ordered pair of letters.

If $n = 3$, for example, such a pair is

$$\begin{array}{ccc} A & B & C \\ B & C & A \\ C & A & B \end{array} \qquad \begin{array}{ccc} \alpha & \beta & \gamma \\ \gamma & \alpha & \beta \\ \beta & \gamma & \alpha \end{array}$$

Greek letters are commonly used in the second square so that the composite square is a " Graeco-Latin " or Euler Square. Euler was unsuccessful in his attempt to solve the problem for $n = 6$: " Or après toutes les peines qu'on s'est donné pour résoudre ce Problème on a été obligé de reconnoître qu'un tel arrangement est absolument impossible, quoiqu'on ne puisse pas en donner de démonstration rigoureuse."

In 1900 G. Tarry furnished a proof that no solution of the problem was possible. He showed that the 6×6 squares could be analysed into 17 types, and that no solution could be found for any of these types. He later solved a similar problem when $n = 8r$ and r is odd and not divisible by 3. It has been shown that a solution is possible if $n = 2^k$, $2p + 1$, $(2p + 1)2^k$. $k > 1$ and it is surmised that no solution is possible if $n = 2(2p + 1)$, but this has been proved only for $p = 0$, 1.

The construction of Latin Squares received an impetus when they were applied to statistical investigations. For example, one way of investigating the differences of yield of n varieties of wheat is to divide the plot into n^2 sub-plots in the form of a Latin Square and associate each variety with one of the n different letters. In other experiments it is required to find sets of mutually orthogonal squares. The greatest number in such a set for squares of side n is $(n - 1)$. Such sets have been constructed for every n from 2 to 13, except 6, 10, 12. No Graeco-Latin square is known to exist when $n = 10$.

For $n = 12$ Graeco-Latin squares exist, but no triple of mutually orthogonal squares has been found.

Two symbolic solutions to the problem of enumeration of Latin squares of any order have been given by MacMahon. The first, which involves the perfect partitions of a number, is admitted by MacMahon himself to become impracticible if n exceeds 5. The other method is concerned with the partitions of multipartite numbers and appears in practice to be equivalent to writing out all possible squares. It is usual to find the number of reduced Latin squares, *i.e.* those in which the first row and column are each written in alphabetical order. Apart from the symbolic solutions, the actual enumeration has been carried out in practice in two ways.

The first method, in which the square is built up line by line, uses as a check on the first three lines the solution of two well-known problems :

1. Derangements (Problème des rencontres). To find the number of ways in which every one of n numbers may be displaced.
2. " Problème des ménages." In how many ways can n husbands sit at a table with their n wives so that no husband is next to his own wife. This requires the finding of the number of permutations discordant with $1, 2, 3 \ldots (n-1), n$

$$2, 3, 4 \ldots \quad n, \quad 1.$$

A solution of the generalised problem (in fact the one required) of finding the number of permutations discordant with two given permutations was given by J. Touchard in 1934.

This method was considered by Cayley and applied by him to $n = 4, 5$ in 1890. Results for $n = 6, 7$ were obtained by Sade in 1948. He shows that Latin rectangles of two lines may be classified according to the permutation which changes the first line into the second, and that the same number of squares result from two rectangles of the same type. A similar result is true for rectangles of any number of lines.

The other method involves considering the symmetries in the squares as a whole. The squares are classified according to these symmetries, and in particular the type of the leading diagonal, and then the enumeration of each class is carried out by finding what transformation of rows, columns and elements will produce a different square without altering the type. This method was used by Fisher and Yates in 1934 for $n = 6$ and by Norton for $n = 7$ in 1939, though as Sade shows, the enumeration of the latter was in error.

The enumeration for n greater than 7 has not yet been effected and, according to Sade, his method might be expected to involve about 30,000 hours' work for the 8×8 square.

The numbers of reduced Latin squares are

$n = 2$	3	4	5	6	7
Number of squares = 1	1	4	56	9,408	16,942,080.

E. J. Armstrong.

2538. *A note on the partial sum of a certain hypergeometric series.*

In the usual notation let

$$ {}_{k+1}F_k(a, a_1, \ldots, a_k ;\ b_1, \ldots, b_k ;\ 1)_N $$

$$ = \sum_{n=0}^{N} \frac{(a_1)_n \ldots (a_k)_n (a)_n}{(b_1)_n \ldots (b_k)_n \ {}_n n!} $$

where $(a)_n = a(a+1)(a+2) \ldots (a+n-1)$.

Let $v_n = (a+n)(a_1+n) \ldots (a_k+n)\alpha_n$.

Then $v_n - v_{n-1} = \alpha_n\{(a+n)(a_1+n) \ldots (a_k+n) - (b_1+n)(b_2+n) \ldots (b_k+n)n\}$.

Suppose now that the terms in powers of n in the brackets vanish. Then

$$a + a_1 + \ldots + a_k = b_1 + \ldots + b_k,$$
$$aa_1 + a_1a_2 + \ldots + a_{k-1}a_k = b_1b_2 + \ldots + b_{k-1}b_k,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$aa_1 \ldots a_{k-1} + \ldots + a_1 \ldots a_k = b_1 \ldots b_k.$$

That is, $-a, -a_1, \ldots, -a_k$ are the roots of the equation

$$x^{k+1} + A_1x^k + A_2x^{k-1} + \ldots + A_{k+1} = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and $-b_1, -b_2, \ldots, -b_k$ are the roots of the equation

$$x^k + A_1x^{k-1} + A_2x^{k-2} + \ldots + A_k = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

It follows that

$$v_n - v_{n-1} = \alpha_n \cdot aa_1a_2 \ldots a_k,$$
$$v_{n-1} - v_{n-2} = \alpha_{n-1} \cdot aa_1a_2 \ldots a_k,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$v_1 - v_0 = \alpha_1 \cdot aa_1a_2 \ldots a_k,$$
$$v_0 = \alpha_0 \cdot aa_1a_2 \ldots a_k,$$

and so

$$v_n = \sum_{r=0}^{n} \alpha_r \cdot aa_1a_2 \ldots a_k,$$

that is

$$ {}_{k+1}F_k(a, a_1, \ldots, a_k;\ 1+b_1, \ldots, 1+b_k;\ 1)_n$$
$$= \frac{(1+a)_n(1+a_1)_n \ldots (1+a_k)_n}{n!\,(1+b_1)_n(1+b_2)_n \ldots (1+b_k)_n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

under conditions (1) and (2).

Further, if n tends to infinity in (3),

$$ {}_{k+1}F_k(a, a_1, \ldots, a_k;\ 1+b_1, \ldots, 1+b_k;\ 1)$$
$$= \frac{\Gamma(1+b_1)\,\Gamma(1+b_2) \ldots \Gamma(1+b_k)}{\Gamma(1+a)\,\Gamma(1+a_1)\,\Gamma(1+a_2) \ldots \Gamma(1+a_k)}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Since $a + a_1 + a_2 + \ldots + a_k = b_1 + b_2 + \ldots + b_k$, this series is convergent for all values of $a, a_1, \ldots, a_k$, and $b_1, \ldots, b_k$.

In particular

$$ {}_2F_1(a, b;\ 1+a+b;\ 1)_n = (1+a)_n(1+b)_n/n!(1+a+b)_n \quad \ldots\ldots\ldots\ldots\ldots(5)$$

and

$$ {}_3F_2(a, b, c;\ d, 2+a+b+c-d;\ 1)_n = \frac{(1+a)_n(1+b)_n(1+c)_n}{n!(d)_n(2+a+b+c-d)_n}, \quad \ldots\ldots(6)$$

where $bc + ca + ab = (d-1)(1+a+b+c-d)$.

L. J. SLATER

2538. *Four-point systems of real conics.*

Many of the demonstrators of the properties of such conics are content to establish them for the case of four real distinct points and allow the main-

tenance of the property in other cases to be assumed by default, although on occasions the method of proof fails. The present paper is an attempt to establish the principal properties by fairly uniform methods and with use only of the real field. It may be remarked that while four distinct points define a system, further assumptions have to be made in cases of coincidence; thus if two points or two pairs of points coincide it is usual to consider that the tangent direction or directions are known, while if three or four points coincide it is also necessary to have knowledge of a non-degenerate conic of the system. In all cases we shall assume the system defined and that, in the first instance, the four points are real with only stated coincidences.

Notation.

We denote the four points by A, B, C, D. We will distinguish between corresponding points attached to conics S_1, S_2 by the suffixes 1, 2.

1. The effects on the harmonic triangle have been treated by the author in a previous paper (where, however, the points were tacitly assumed real).*

2. *Desargues' Theorem.*

Case 1. A, B, C, D distinct. Project AB to be the line at infinity obtaining a system of hyperbolas with parallel asymptotes. Suppose a given fixed line, not through a pivot point, meets CD in O and meets S in P, Q and its asymptotes in p, q: let CD meet the asymptotes in c, d, p and c being on the same asymptote. Then

$$\frac{OP \,.\, OQ}{OC \,.\, OD} = \frac{Op \,.\, Oq}{Oc \,.\, Od}.$$

Further, for conics S_1, S_2, from the parallelisms of the asymptotes,

$$\frac{Op_1}{Oc_1} = \frac{Op_2}{Oc_2}, \quad \frac{Oq_1}{Od_1} = \frac{Oq_2}{Od_2};$$

$$\therefore \; OP_1 \,.\, OQ_1 = OP_2 \,.\, OQ_2,$$

and P, Q are pairs of an involution of which the intersections of the line with AB and CD form a pair.

Case 2. C, D coincide. The chord CD of case 1 becomes a tangent.

Case 3. C coincides with A and D with B. The projected conics have common asymptotes: the mid-point of PQ coincides with that of pq and the involution is established.

Case 4. A, C, D coincide. Projecting the tangent at A to be the line at infinity the conics become parabolas with parallel axes and meeting in one finite point, B, only. The parabolas are therefore equal and turned in the same direction. Then $OP \,.\, OQ = f \,.\, BO$ where O is the point where the diameter through B meets the given line and f is the length of the parallel focal chord. But f is constant and hence $OP \,.\, OQ$ is constant.

Case 5. A, B, C, D coincide. With the previous projection the axes of the equal parabolas coincide. The mid-point of PQ is fixed and the involution is again established.

3. *The polars of a fixed general point and the poles of a fixed general line.*

Cases 1, 2, 3, 4, A and B being distinct points. Let P be a general point with PA, PB meeting S in A', B' and its polar, p, with respect to S meeting them in α, β. Then at once $\{A'\} \overline{\wedge} \{B'\}$ and therefore $\{\alpha\} \overline{\wedge} \{\beta\}$. Furthermore, the homographies have as common point the point P (given when S passes

* *Math. Gaz.*, Feb. 1944, Vol. XXVIII.

through P). Hence the polars of a fixed general point pass through a fixed point, P'.

Taking points P, Q on a fixed general line, we have $P'\{p\} \overline{\wedge} Q'\{q\}$, and the locus of the pole of the line is a conic.

Case 5. The points all coinciding. With the projection of paragraph 2, Case 5, the polars of P are parallel, passing through a fixed point P' on the line at infinity (the tangent at the common point of the conics). Furthermore, the homography $P'\{p\} \overline{\wedge} Q'\{q\}$ has this tangent as a common line and the locus of the pole of PQ is a straight line. It may be noticed that the same figure gives at once the well-known theorem for this system of conics that any point P' on the tangent at the common point has the same polar for all the conics.

4. *The harmonic envelope and the harmonic locus for two conics S_1, S_2.*

Project any line through a general point X and not a tangent to S_1 or S_2 into the line at infinity, so that the conics become central conics. In the projected figure let the polar of X w.r.t. S_1 meet S_1 in U, U' and meet S_2 in V, V'. Through any point Y on UU' draw XY (one of a set of parallel straight lines) to meet S_1 in P, P' and S_2 in Q, Q'. Then for all positions of Y, $P'Y = YP$ and, from the constancy of the direction of XY, $YP^2/U'Y \,.\, YU$ is a constant and $Q'Y \,.\, YQ/V'Y \,.\, YV$ is a constant. From the harmonic condition $YP^2 = Q'Y \,.\, YQ$ and $U'Y \,.\, YU/V'Y \,.\, YV$ is therefore constant and there are two satisfying positions of Y and the envelope is therefore of class two. The envelope clearly touches the tangents to S_1, S_2 at their common points. If S_1 is degenerate, U, U' coincide at the meeting point of the two lines: neither the degeneracy of S_1 or/and of S_2 affects the proof. The method however does not display the special cases which are listed in the author's paper already referred to.

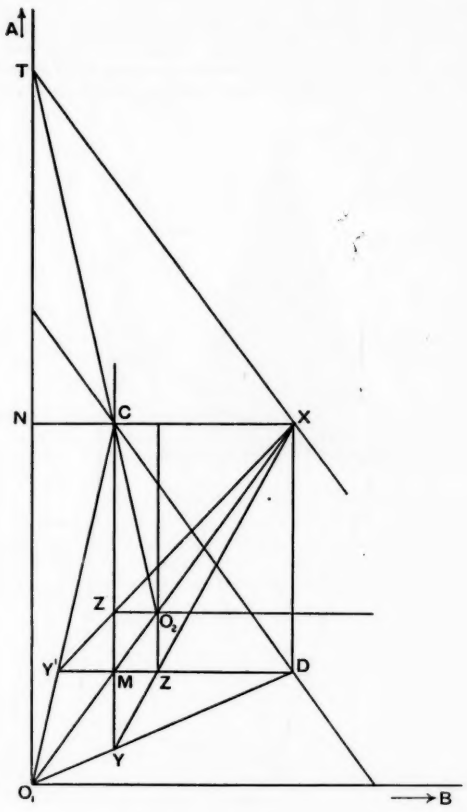
The property of the harmonic locus can now be obtained by reciprocation.

5. *The theorem.* "If, with distinct intersections, the pole of AB w.r.t. S_1 coincides with the pole of CD w.r.t. S_2, then the pole of AB w.r.t. S_2 coincides with the pole of CD w.r.t. S_1, and the harmonic envelope degenerates to these poles, the conics being non-degenerate."

Project the line AB to the line at infinity with AB subtending a right-angle at any finite point to render the conics rectangular hyperbolas with parallel asymptotes. S_1 has centre O_1, and asymptotes O_1A, O_1B and passes through C, D: S_2 touches O_1C, O_1D at C, D. Using the Pascal hexagon $AACBDD$ inscribed in S_2, AD and BC give X, AC and DD (DO_1) give Y, the intersection of XY with DB gives the third Pascal point, Z, and hence the asymptote AA through Z. From the Pascal hexagon $BBDACC$ the other asymptote BB through Z' is determined and hence also the centre O_2, as shown in the figure. Since the mid-point of the chord CD of both conics is also the mid-point of the intercepts between both pairs of asymptotes, the two rectangles formed by the pairs of asymptotes and XC, XD have a common diagonal with the rectangle $CXDM$. The Pascal hexagon $AABCCD$ inscribed in S_1 gives as axis the line through X parallel to CD, and if this axis meets O_1A in T the tangent to S_1 at $C(CC)$ is CT. But $Z'O_2/Y'M = XO_2/XM = CZ'/CM$, and therefore $\angle Z'CO_2 = \angle MCY'$. Furthermore if XC meets O_1A in N, $O_1T = 2O_1N$ and hence T, C, O_2 are collinear and O_2C is the tangent to S_1 at C. Similarly O_2D is the tangent to S_1 at D and O_2 is the pole of CD w.r.t. S_1. There are no special cases of value.

From the uniqueness of the construction it follows that for each S_1 of a distinct four-point system there is one and only one S_2 satisfying the condi-

tions, and also, using the coincidence of the mid-points of CD and its asymptote intercepts, that "If non-degenerate conics S_1, S_2 intersect in A, B, C, D and are such that the chord CD meets the tangents at A, B to S_1, S_2 in pairs of points forming an involution with C, D, there is a corresponding property for the chord AB."



6. We will now suppose that A, B are conjugate imaginary points: AB is then a line of the real degenerate conic. Take P, P' and Q, Q' two real pairs of points on AB which are conjugate points with respect to a conic of the system: they are pairs of an elliptic involution and PP', QQ' overlap. Projecting AB to be the line at infinity in such a way that PP' and also QQ' subtend right angles at any finite point the conics now have two pairs of perpendicular conjugate directions and are therefore circles. The system becomes a coaxal system of circles in Cases 1, 2, and a concentric system of

circles in Case 3 (the other cases do not arise). The properties of paragraph 1 to 3 are then standard text-book exercises : simple proofs of those of paragraphs 4 and 5 are incorporated in the author's previous paper.

H. GWYNEDD GREEN

2539. *The dual polyhedral compound : Great Icosahedron + Great Stellated Dodecahedron.* $\{3, 5/2\} + \{5/2, 3\}$.

Of the nine regular polyhedra, one (the tetrahedron) is self-dual, and the remainder can be grouped in dual pairs. Each dual pair can be placed together in such a way that they are polar reciprocals with respect to a sphere touching their edges ; the edges of each polyhedron then bisect those of the

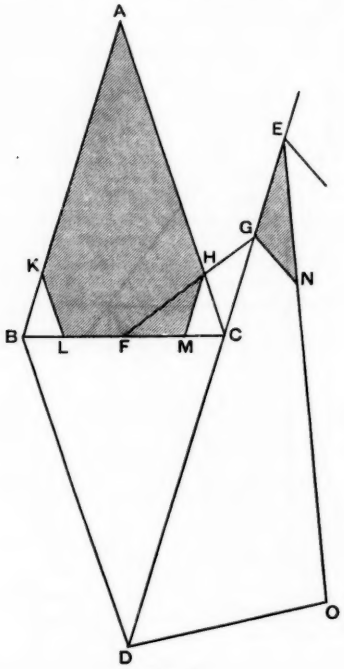


FIG. 1.

$$AB = AC = BD = CD, \quad \angle BAC = 36° ;$$
$$BF = FC = CG = GE = FH, \quad BK = KL = MH = HC = HG = GN ;$$
$$DO = BC ; \quad \angle CGN = 60° = \angle CDO, \quad \angle GEN = 22° 14'.$$

other at right angles. Five dual compounds can thus be constructed : (1) two tetrahedra ; (2) cube + octahedron ; (3) dodecahedron + icosahedron ; (4) great dodecahedron + small stellated dodecahedron ; and (5) great icosahedron + great stellated dodecahedron. I have elsewhere given details of the construction of the first four of these (Cundy and Rollett, *Mathematical*

Models, p. 124). The purpose of this note is to describe the construction of the fifth.

Mr. Dorman Luke of Florida very kindly sent me details of a similar compound made by him, also described below, which has the same edges as (5). From this it was easy to derive the net for the construction of the dual compound itself.

Fig. 1 shows in a single diagram parts of the faces of the great stellated dodecahedron and the great icosahedron in their correct proportion. The external elements of the net of the dual compound are shaded. Fig. 2 shows

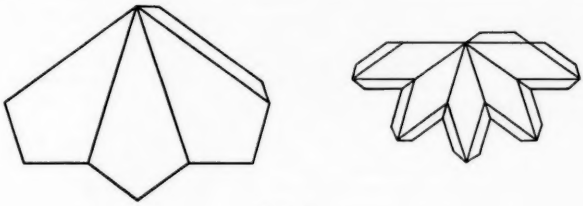


FIG. 2.

A

20 of these required. Score internal lines before folding.

B

12 of these required. Score full lines on front, dotted lines on back.

the two different units of the net. The compound can be constructed from 20 *A* units and 12 *B* units. In fact, I made the complete great stellated dodecahedron from 20 trihedra, and stuck on the 12 *B* units in the depressions between its points.

Fig. 4 shows the corresponding faces of Mr. Luke's compound. This is the pair, great stellated dodecahedron + small stellated dodecahedron, which has the same edges but the planes and vertices are not reciprocal. It makes an effective model in two colours. The faces of the basic dodecahedra are shaded, and also the external elements of the net of the two stellates. Fig. 3

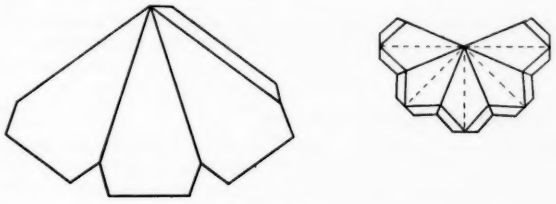


FIG. 3.

A

20 of these required

B

12 of these required.

shows the two different units of the net; again 20 *A* units and 12 *B* units are needed, and a very satisfactory model can be made in this way. It is an interesting fact that if either model is held with two trihedral vertices uppermost in a horizontal line, the remaining vertices of the compound fall into horizontal planes containing 6, 4, 8, 4, 6, 2 vertices respectively. Reciprocally,

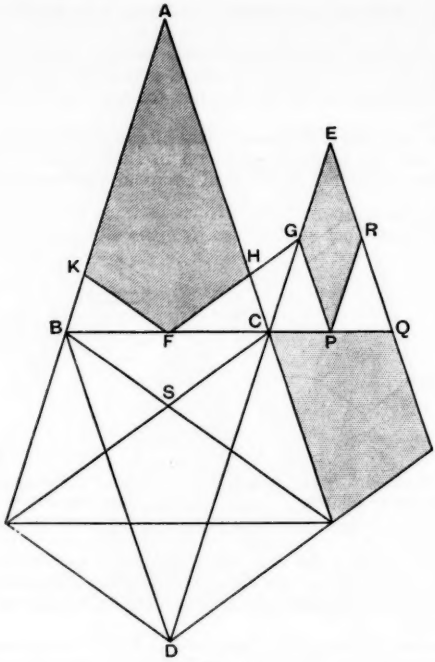


FIG. 4.

Letters as in Fig. 1. $CP = PQ = \frac{1}{2}CS$.
Core-face and element of great stellated dodecahedron hatched.
Core-face and element of small stellated dodecahedron stippled.

the planes of the faces meet by 6, 4, 8, 4, 6 in points of a vertical line, which the reader (with the model before him) will be able to identify.

H. MARTYN CUNDY.

2540. *On even distribution of numbers.*

In that classic of the game, *Why You Lose at Bridge*, the lamented S. J. Simon tells how he discovered that the best way to deal with a bid which he suspected of being an attempt to bluff him was to treat it as if it was genuine. I am sure Mr. Hope-Jones is far too old a bird to have been caught in the snare in which he invites us (Note 2343, XXXVII, p. 203) to watch him fluttering, but for the benefit of fledgling readers I propose to take him at his word. The solution of his puzzle is in an obvious question : Is the number of numbers with which we are dealing finite or infinite?

In an infinite field, any estimate of the ratio of a part to the whole by enumeration is nonsense ; none of us to-day reads paradoxes into theorems of the familiar type that the number of the integers 1, 2, 3, ... is the same as the number of the even integers 2, 4, 6, ... An estimate by integration embodies, as Mr. Hope-Jones points out, a postulate of uniformity which may be chal-

lenged ; he makes no secret of the fact that equiprobability of logarithms is as blatant an assumption as equiprobability of values.

In a finite field, there are no fallacies or paradoxes, but only questions to which we do not know the answers. The available numbers belong to some definable class of which we are tacitly supposing that as many of its members begin with any one digit as with any other ; if more of the available numbers begin with 5 than with 6, a choice in which the selected number is as likely to begin with 6 as with 5 can not be in any sense a random choice.

The simplest forms of uniform class are the class which consists of all the numbers with not more than a given number of digits, and the class which consists of all the numbers with precisely a given number of digits. An understanding that the available numbers form a class of one of these two kinds and that all the members of the class are used the same number of times is beneath most arguments about the relative probabilities of digits. If x is a variable member of a well-defined uniform class, the question whether for a given function the class of numbers $f(x)$ is itself uniform is a definite mathematical problem which can hardly be trivial.

We know that the leading digits and the terminal digits of $2x$ and x^2 are subject to limitations, and that the leading figures in a table of $\sin x$ to an assigned number of places are not distributed uniformly if the interval in x is constant. Nevertheless, the orthodox view is that if we cut off a figure or two from the beginning and half a dozen figures from the end of every entry in any mathematical table, we can expect uniformity of distribution if the table is sufficiently extensive. How a question which is not formulated precisely can be supposed to have a precise answer I do not know, but what is this expectation but a superstition which we ought to have outgrown, that in cases of complete ignorance we have some transcendental assurance that any one answer is as likely to be right as any other answer? We do not know if there is a red dragon on the far side of the moon, but we do know that if there is one, it has a yellow tail with 137 green spots. E. H. N.

2541. *On digital distribution.* (See Note 2343, XXXVII, No. 321, p. 203.)

If numbers are evenly distributed, the first significant digits of their representation in radix notation are not evenly, but logarithmically, distributed. In radix r, the proportion of number representations beginning with the digit d will be $\log_r \{(d+1)/d\}$.* For instance, it is evident that the binary representation of all numbers, other than zero, must begin with " 1 ".

This logarithmic distribution of leading significant figures has been noticed, particularly in statistical tables, for a long time. Usually it has been interpreted as a law of nature, and as implying that natural magnitudes are distributed logarithmically. These views are prevalent in the world of mechanized computing which, possibly because of recent heavy infiltration by electronic engineers, is currently riddled with mathematical superstitions.

So far from its being a consequence of the way in which numbers occur it is, as Mr. Hope-Jones demonstrates, a consequence of the way we talk about them.

More interesting than the logarithmic distribution itself is its insensitivity to changes in the distribution of numbers. This has been studied in detail by Furry and Hurwitz,† who show, *inter alia*, that the logarithmic relation holds good whenever, and to the same extent as, a certain integral can be replaced by a certain sum. R. A. FAIRTHORNE.

* S. A. Goudsmit and W. H. Furry. " Significant Figures of Numbers in Statistical Tables," *Nature*, 154, pp. 800–1, December 23, 1944.

† W. H. Furry and H. Hurwitz, " Distribution of Numbers and Distribution of Significant Digits," *Nature*, 155, pp. 52–3, January 13, 1945.

2542. *Notes on conics.* 19: *The asymptotic involution of the ellipse.*

If the factors of $ax^2 + 2hxy + by^2$ are real, the equation

$$ax^2 + 2hxy + by^2 = 0$$

represents a pair of lines, and the equation

$$ax^2 + 2hxy + by^2 = 1 \quad \ldots\ldots(1)$$

expresses that the product of the distances of the point (x, y) from these two lines has the constant value $1/\sqrt{\{(a-b)^2 + 4h^2\}}$. What does the equation (1) express directly, in real terms, if the quadratic does not factorise? More precisely, what is the relation of the equation (1) to the *involution*

$$ax_1x_2 + h(x_1y_2 + y_1x_2) + by_1y_2 = 0 \quad \ldots\ldots(2)$$

in a geometry where double lines must not be postulated?

Consider the involution determined on the circle

$$x(x-p) + y(y-q) = 0 \quad \ldots\ldots(3)$$

by the variable line

$$lx + my = lX + mY$$

through the fixed point (X, Y). The line-pair joining the origin to a point-pair of this involution is

$$(lX + mY)(x^2 + y^2) = (lx + my)(px + qy),$$

that is, $(x_1y - y_1x)(x_2y - y_2x) = 0$, where

$$\frac{x_1x_2}{lX + m(Y-q)} = \frac{x_1y_2 + y_1x_2}{lq + mp} = \frac{y_1y_2}{l(X-p) + mY},$$

and therefore the line-pair belongs to the involution (2) if

$$Xa + qh + (X-p)b = 0, \quad (Y-q)a + ph + Yb = 0.$$

In other words, the pole of the involution determined on the circle (3) by the line-involution (2) is given by

$$(a+b)X = bp - hq, \quad (a+b)Y = -hp + aq,$$

implying

$$(a+b)(X-p) = -ap - hq, \quad (a+b)(Y-q) = -hp - bq.$$

It follows that the axis of the involution on the circle, which is the polar of (X, Y) and can be written

$$(X-p)x + X(x-p) + (Y-q)y + Y(y-q) = 0,$$

has the equation

$$(ap + hq)x + (hp + bq)y - (bp - hq)(x-p) - (-hp + aq)(y-q) = 0,$$

and the distance of (p, q) from this line is

$$(ap^2 + 2hpq + bq^2)/\sqrt{[\{(a-b)p + 2hq\}^2 + \{2hp - (a-b)q\}^2]}$$

that is,

$$\frac{ap^2 + 2hpq + bq^2}{\sqrt{\{(a-b)^2 + 4h^2\}} \cdot \sqrt{(p^2 + q^2)}}.$$

That is to say, if a circle is described on the line joining the origin to (p, q) as diameter, the normalised expression

$$(ap^2 + 2hpq + bq^2)/\sqrt{\{(a-b)^2 + 4h^2\}} \quad \ldots\ldots(4)$$

measures the product of the distance of (p, q) from the origin and the distance of (p, q) from the axis of the involution determined by (2) on the circle.

This interpretation of (4) is real whether or not the line-involution has real double lines, and it is only what we should expect. If the double lines are real, they meet the circle in two points E, F, the axis of the involution on the circle is the line EF, and the product of the distances of (p, q) from the origin and from EF is equated to the product of the distances of (p, q) from the two points E, F, that is, to the product of the distances of (p, q) from the two double lines, by the theorem that used to be known as Euclid, VI, C. As in almost all interpretations along v. Staudt's lines, we know the answer before we attempt to justify it in the field in which we pretend to be working.

E. H. N.

2543. *Ternary block systems and pentatonic packaging.*

The results here discussed should admit of application in standardisation of packaging. They might also interest architects or designers concerned with rectangular sub-divisions of spaces.

For r given, we use the symbol (x, y, z) to represent a rectangular block of dimensions $r^x \times r^y \times r^z$, and assume x, y, z all integral.

Consider now three particular sets of blocks for the case $r = 2^{1/5}$, each containing three blocks as below, together with all the other blocks derived by making equal integral changes in all three symbols in each bracket :

Set 1 : typical blocks $(0, 6, 7)$, $(0, 1, 7)$, $(0, 1, 2)$;
Set 2 : typical blocks $(0, 4, 6)$, $(0, 2, 6)$, $(0, 6, 12)$;
Set 3 : typical blocks $(0, 7, 9)$, $(0, 2, 9)$, $(0, 7, 14)$.

Each set then has volumes increasing in geometric progression with common ratio r, and identical shapes recur at intervals of three steps, with volumes for each of the three shapes increasing in geometric progression with common ratio r^3. Sets of this type, for any r, will be called *ternary block systems*.

For the sets given, we have $r = 1{\cdot}149$, $r^3 = 1{\cdot}516$; we note also that $r^5 = 2$, $r^8 = 3{\cdot}031 \simeq 3$, $r^{14} = 6{\cdot}964 \simeq 7$.

If we ignore the one per cent excess of r^8 over 3, we can then establish the following results :

(1) Consider any pair of these sets, Set A and Set B, say, $A = 1$, 2 or 3, $B = 1$, 2 or 3, with $A = B$ or $A \neq B$, and let n be any integer $2^R 3^S$, with R, S integral : then if $N = ABn$, a number N of any block in either set may be packed into the size and shape of some block in the other set. For $A = B = 2$, N may also be any power of 3, and for $A = B = 3$, N may also be any power of 2.

(2) Twelve blocks will pack in their own shape for any block in Set 1 or Set 2 : similarly for eighteen of any block in Set 1 or Set 3. No other shapes have these same properties. In any set, there are only four distinct shapes of block faces, and for Sets 1 and 2 together (or 1 and 3), there are only six.

Advantages in packaging will be apparent. Waste space for permitted multiples is only one per cent for each factor 3, and zero for factors 2. For Sets 1 and 2 singly or in combination, successive factors 3 are distributed cyclically to different edges of the assembly.

If we omit alternate blocks in Set 2 we have a set, Set 2′ say, with volume ratio 1·320 for the set and 2·297 for each sub-set, such that inter-packability occurs for any $N = 3^R 4^S$, with R, S integral. Permitted multiples 3, 4, 9, 12, 16, 27, 36, 48, 64, 81, 108, 144, ... are numerous and conveniently spaced. This set seems worthy of special notice.

In place of a strictly geometric series for $r = 2^{1/5}$, we may use the set of numbers 12, 14, 16, 18, 21 continued by successive doubling of all five terms. This gives an *integral** series at the price of some concentration of discrepancies. Analogies with pentatonic scales in music will be obvious.

In the specification of external and internal dimensions respectively for individual and multiple containers, with provision for clearance, we may use integral series based in similar fashion on two neighbouring sets of numbers, such as for instance 40, 46, 53, 61, 70 and 41, 47, 54, 62, 71.

The preceding results include particular applications of the following general theorems :

(A) For factors $h \times k \times l$ of M, we can find two shapes of block, with edges inversely proportional, which will allow M blocks to pack in their own shape by $h \times k \times l$ assembly, if h, k, l are distinct, but only one shape, with edges in geometric progression, if two of h, k, l are equal, apart from special cases where M is a perfect cube with h, k, l either all equal or in geometric progression.

(B) For factors $h \times k \times l$ of M and $u \times v \times w$ of N, sets of three shapes of block can be found such that each packs in the shape of some one of the other two (either the *same* one, in a *direct* set, say, or *different* ones, in an *inverse* set, say) in assemblies by $h \times k \times l$ and by $u \times v \times w$ respectively. In general cases there are eight distinct sets for given h, k, l and u, v, w. This number is reduced if there are equal or geometrically progressing factors of M or N, or directly or inversely proportional factors of M and N.

Proofs of these theorems involve only a substantial amount of quite elementary algebra.

For versatility of packing, the most interesting sets of factors are of the form $1 \times 1 \times a$, $1 \times 1 \times b$, with $a \neq b$. Here there are unique and distinct direct and inverse sets, each containing two blocks with inversely proportional edges and one block with edges in geometrical progression.

To avoid having a *two-parameter* series of blocks, we must accept a little waste space for one factor and use an approximate closure relation of the form $a^q \simeq b^p$. Then if $p - q$, for direct sets, or $p + q$, for inverse sets, is a multiple of 3, one block for each volume suffices, and the series of individual edge lengths which are used has the same common ratio r as the series of volumes. In fact, here and only here we have a *ternary block system*.

With $r = 2^{1/5} \simeq 3^{1/8}$ we have 5, 8, 10 steps, respectively, for multiplications by 2, 3, 4. Direct ternary sets for $M = 2$, $N = 3$ and inverse ternary sets for $M = 3$, $N = 4$ are then possible, whence we derive Sets 1, 2, 2′.

Other choices of sets are possible, but probably less convenient. The best set for decimal* multiples seems to be the inverse set for $a = 4$, $b = 5$, based on $r = 5^{1/8} \simeq 4^{1/7}$. Typical blocks here have symbols (0, 2, 5), (0, 3, 5). (0, 5, 10). Permitted multiples are $N = 4^R 5^S$, with R, S integral, namely 4, 5, 16, 20, 25, 64, 80, 100, 125, 256, 320, 400, 500, ..., and packing is with unaltered shape for 20, 64, 125, 400, ...

Here $r = 1{\cdot}223$ and $r^3 = 1{\cdot}829$. Suitable integral series are given either by the series obtained from 18, 22, 27, 33, 40, 50, 60, 72, or by the series obtained from 20, 24, 30, 36, 45, 55, 66, 81, with continuation by quintuplications. (There are analogies here with diatonic musical scales.) For each factor 4 there is about 2 per cent of waste space ; successive factors 4 are distributed cyclically to different edges.

THOS. H. O'BEIRNE.

* Sears, J. E., *Report on the selection of ranges of types and sizes (preferred numbers)*, B.S. 1638: 1950 (London, British Standards Institution, 1950).

2544. *The expansion of polynomials in sin θ and cos θ as series of associated Legendre functions of cos θ.*

1. Formulae for $\cos n\theta$, $\sin\theta$ and $\sin n\theta$ in terms of ordinary Legendre polynomials are well known, though none of them is simple in form and only the first leads to a finite series. The first is given in a general form and also with numerical results in N. M. Ferrers' *Spherical Harmonics* (1877), pp. 30–5, and in T. M. MacRobert's *Spherical Harmonics* (1927), p. 106, Exx. 14, 15. The second is given by Ferrers, pp. 35, 36, and all three are given in W. E. Byerly's *Fourier Series and Spherical Harmonics*, pp. 182–4. It is remarkable that these formulae are more complicated than the two fundamental formulae given below for associated Legendre functions. These formulae may be used to expand any polynomial in $\sin\theta$ and $\cos\theta$ in terms of associated Legendre functions. As far as can be ascertained, these formulae have not been given explicitly in any of the literature belonging to the subject. In particular, they are not found in E. W. Hobson's *Spherical and Ellipsoidal Harmonics* or in Heine's *Kugelfunktionen*, but it has been pointed out by Professor T. M. MacRobert that both formulae may be derived from a result in his *Spherical Harmonics*.

2. The two formulae are

$$\cos n\theta = P_n^0 - \frac{2P_n^2}{(n+1)(n+2)} + \frac{2P_n^4}{(n+1)(n+2)(n+3)(n+4)} - \dots$$

$$= P_n^0 - 2\,.\,n!\sum_{k=1}^{m}\frac{(-)^k P_n^{2k}}{(n+2k)!}; \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\text{(i)}$$

$$\sin n\theta = \frac{2P_n^1}{n+1} - \frac{2P_n^3}{(n+1)(n+2)(n+3)} + \frac{2P_n^5}{(n+1)(n+2)(n+3)(n+4)(n+5)} - \dots$$

$$= 2\,.\,n!\sum_{k=0}^{m}\frac{(-)^k P_n^{2k+1}}{(n+2k+1)!}. \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\text{(ii)}$$

where $x=\cos\theta$, $y=+\sqrt{(1-x^2)}=\sin\theta$ and $0\leqslant\theta\leqslant\pi$,

and $$P_n^k \equiv P_n^k(\cos\theta) = y^k\left(\frac{d}{dx}\right)^k P_n(x).$$

In (i) m is $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$ according as n is even or odd;
in (ii) m is $\frac{1}{2}(n-2)$ or $\frac{1}{2}(n-1)$ according as n is even or odd.

With a slight change of notation both formulae can be obtained by putting $\phi=0$ in formula (20) on p. 128 of MacRobert's book,

$$\{x+\sqrt{(x^2-1)}\,.\cos\phi\}^n = P_n(x) + 2\sum_{m=1}^{n}\frac{n!}{(n+m)!}\cos m\phi\,.\,P_n^m(x).$$

An elementary proof for both (i) and (ii) may also be obtained as follows. From the known forms of P_n^k, $\cos n\theta$ and $\sin n\theta/\sin\theta$, we can assume that

$$\cos n\theta = A_0P_n^0 + A_2P_n^2 + A_4P_n^4 + \dots,$$

$$\sin n\theta = A_1P_n^1 + A_3P_n^3 + A_5P_n^5 + \dots.$$

The use of the results

$$\frac{d}{dx}(\sin n\theta) = -\frac{n}{y}\cos n\theta, \quad \frac{d}{dx}(\cos n\theta) = \frac{n}{y}\sin n\theta$$

and the recurrence relation

$$(mx/y)\,.\,P_n^m = \tfrac{1}{2}\left\{P_n^{m+1} + (n-m+1)(n+m)\,P_n^{m-1}\right\}$$

leads to the two sets of recurrence relations

$$\left.\begin{aligned} 2nA_1 &= 2A_0 - (n-1)(n+2)A_2, \\ 2nA_3 &= A_2 - (n-3)(n+4)A_4, \\ 2nA_5 &= A_4 - (n-5)(n+6)A_6, \\ &\cdots\cdots\quad\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots(\alpha)$$

$$\left.\begin{aligned} 2A_0 &= (n+1)A_1, \\ -2nA_2 &= A_1 - (n-2)(n+3)A_3, \\ -2nA_4 &= A_3 - (n-4)(n+5)A_5, \\ &\cdots\cdots\quad\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots(\beta)$$

The special case $\theta = 0$, $x = 1$, $y = 0$ gives $A_0 = 1$, and the formulae (i) and (ii) are obtained immediately.

3. It is clear that the two fundamental formulae may be used to express any polynomial in $\sin\theta$ or $\cos\theta$, or $\sin\theta$ and $\cos\theta$ together, in terms of associated Legendre functions of $\cos\theta$, though the general expressions of these results are heavy and almost useless as formulae. Since a general polynomial is of the form $\Sigma a_{mn} \cos^m\theta \sin^n\theta$, each term may be expressed as $\Sigma b_k \cos k\theta$ or $\Sigma c_k \sin k\theta$, where k is an integer, according as n is even or odd, and then formulae (i) and (ii) may be applied. One simple example is given here. Taking $m = 4$, $n = 3$,

$$2^7 i^3 \cos^4\theta \sin^3\theta = (z + z^{-1})^4 (z - z^{-1})^3, \quad \text{where} \quad z = e^{i\theta},$$

so that

$$-64\cos^4\theta\sin^3\theta = \sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin\theta.$$

Hence

$$-32\cos^4\theta\sin^3\theta$$
$$= 7!\left\{\frac{P_7^1}{8!} - \frac{P_7^3}{10!} + \frac{P_7^5}{12!} - \frac{P_7^7}{14!}\right\} + 5!\left\{\frac{P_5^1}{6!} - \frac{P_5^3}{8!} + \frac{P_5^5}{10!}\right\} - 3\,.\,3!\left\{\frac{P_3^1}{4!} - \frac{P_3^3}{6!}\right\} - 3\,.\,2!\,\frac{P_1^1}{2!}\,.$$

F. UNDERWOOD and G. POWER.

2545. *Products of perspectives.*

In a recent text-book on projective geometry it was stated (following a previous book on the subject) that the condition for the product of two perspectives in a plane to be a perspective is that the three bases should be concurrent. This is not correct; the condition given is sufficient but not necessary.

If in the plane any point Q on a fixed line b is projected from a fixed point N to a point P on a fixed line a and thence to R on c from a fixed point M, the problem is to find when Q, R are in perspective from a fixed point L. This problem is obviously closely connected with Steiner's problem: to draw a triangle PQR whose vertices lie on three fixed lines a, b, c and whose sides pass through three fixed points L, M, N. The cases in which there are an infinite number of solutions are solutions of the former problem.

There are two cases, both self-dual.

(i) The sides MN, NL, LM of the triangle LMN pass through the vertices bc, ca, ab of the triangle abc.

(ii) L, M, N are in line and a, b, c are concurrent in a point O; but in this case there is also a necessary relation between the elements, namely, OL, OM, ON must be mates of a, b, c in an involution; or if LMN passes through O the range $(OLMN)$ and the pencil (OL, a, b, c) must be projective.

From (i) it follows that if a range (Q) on b is projected from N to (P) on a, which is then projected from M to (R) on c, then if MN passes through bc the product of the perspectives is a perspective from (Q) to (R) with centre L; and also LM, LN pass through ab, ac. It may also be inferred from (i) that if only two of the sides, say LM, LN pass through the corresponding vertices $C=ab$, $B=ca$, the solution of Steiner's problem is not possible unless the third side MN passes through the remaining vertex $bc=A$, for otherwise the two solutions are the degenerate triangles MBB, NCC, where the side BB is interpreted as NL.

In case (ii), when applied to perspectives the additional conditions are satisfied automatically and if only a, b, c are concurrent the product of a perspective from (Q) on b to (P) on a and then to (R) on c is a perspective from (Q) to (R).

It may be noted that it is not necessary for the perspectives to be all in one plane, for in case (ii) it is only necessary for a, b, c to be concurrent, not coplanar; but in case (i) the figure is in the plane LMN.

The synthetic proofs of these results are not difficult, and they can also be verified algebraically. H. V. MALLISON.

2546. *Projectivity on a conic.*

If $(P)\overline{\wedge}(P')$ on a conic s then PP' envelops a conic.

Few text-books give a direct synthetic proof of this theorem; the following, given in two recent books, is such a proof.

Suppose M, N are the united points of the projectivity, so that MN is the cross-axis. Let A, A' be a fixed pair of corresponding points; let PP', AA' meet at R, and PA', $P'A$ meet at Q on MN. Then Q, R are diagonal points of $PP'AA'$ and therefore conjugate points for the conic. Hence $(R)\overline{\wedge}(Q)$. Similarly if PP' meets the join BB' of another fixed pair of corresponding points at R', we have $(R')\overline{\wedge}(Q)\overline{\wedge}(R)$. Hence $RR'\equiv PP'$ envelops a conic t touching AA', BB'.

The object of this note is to give a simple synthetic proof of the fact that s, t touch at M, N.

Let AA', PP' meet MN at C, X. Then Q is a double point of the involution (X, C), (M, N). Hence when X is given on MN, the tangents from X to the envelope of PP', that is, the lines PP' which pass through X, are given by finding the double points of the involution (X, C), (M, N); and thus as the involution has two double points there are two lines through X. But when X is at M or N the involution (M, C), (M, N) or (N, C), (M, N) is degenerate, having a fixed point which is also the sole double point, and there is only one tangent to the envelope t through M or through N, and these are the tangents to s at M or N.

If M, N coincide it appears necessary to appeal to a continuity argument to show that s, t have four-point contact at M. H. V. MALLISON.

2547. *The coincidence locus and envelope.*

In a general correlation of a plane into itself the points which lie on their corresponding lines lie on the coincidence conic-locus and the lines touch the coincidence conic-envelope (surely better names would be *incidence* locus and envelope). It is not difficult to prove geometrically that these two conics have double (or four-point) contact. A simple algebraical proof of this result can be given by means of matrices, adapting to non-symmetrical matrices the elegant method developed in Todd's *Projective and Algebraic Geometry*, Ch. VII.

If the correlation is $\mathbf{u}=\mathbf{a}\mathbf{x}$ where $\mathbf{u}$ and $\mathbf{x}$ are column-vectors of a line, and a point, and $\mathbf{a}=(a_{rs})$ is a 3×3 non-symmetrical matrix, then the coincidence-

locus is $\mathbf{x}'\mathbf{a}\mathbf{x}=0$, and the coincidence-envelope is $\mathbf{u}'\mathbf{A}'\mathbf{u}=0$, where $\mathbf{x}'$, $\mathbf{u}'$ are the row-vectors of $\mathbf{x}$, $\mathbf{u}$, and $\mathbf{A}'$ is the transpose of the adjugate of $\mathbf{a}$, so that $\mathbf{A}'=\mathbf{a}^{-1}/|\mathbf{a}|$.

These represent the conics

$$\mathbf{x}'(\mathbf{a}+\mathbf{a}')\mathbf{x}=0, \qquad \mathbf{u}'(\mathbf{A}'+\mathbf{A})\mathbf{u}=0,$$

where $\mathbf{a}'$, $\mathbf{A}$ are the transposes of $\mathbf{a}$, $\mathbf{A}'$.

The matrix of the associated envelope (that is, of the tangential equation) of the locus is $(\mathbf{a}+\mathbf{a}')^{-1}$, which is a scalar multiple of adj $(\mathbf{a}+\mathbf{a}')$; and to show that this and the coincidence-envelope have double contact it is required to show that there is a number k such that

$$k\,(\text{adj}\,\mathbf{a}+\text{adj}\,\mathbf{a}')-\text{adj}\,(\mathbf{a}+\mathbf{a}') \qquad \text{.....(i)}$$

is the matrix of a conic-envelope reducing to a repeated point, that is, is symmetrical and of rank 1.

Since the cofactors of the elements of $\mathbf{a}$ are of the second degree in the elements of $\mathbf{a}$, we have

$$\text{adj}\,(\mathbf{a}+\lambda\mathbf{a}')=\mathbf{A}+\lambda\mathbf{C}+\lambda^2\mathbf{A}', \qquad \text{.....(ii)}$$

where $\mathbf{C}$ is a 3×3 matrix. Taking $k=2$ in (i) and $\lambda=1$ in (ii), we have, in (i),

$$2(\text{adj}\,\mathbf{a}+\text{adj}\,\mathbf{a}')-\text{adj}\,(\mathbf{a}+\mathbf{a}')=\mathbf{A}-\mathbf{C}+\mathbf{A}'=\text{adj}\,(\mathbf{a}-\mathbf{a}'),$$

from (ii) with $\lambda=-1$. Since $\mathbf{a}-\mathbf{a}'$ is a skew-symmetrical 3×3 matrix, its adjugate is symmetrical and of rank 1, and the required result follows, and the coincidence-locus and coincidence-envelope have double contact.

The pole of the chord of contact has tangential equation

$$\mathbf{u}'\,\text{adj}\,(\mathbf{a}-\mathbf{a}')\mathbf{u}=0,$$

and so it is the point

$$(a_{23}-a_{32},\ a_{31}-a_{13},\ a_{12}-a_{21}).$$

The method cannot be applied to space-correlations. H. V. MALLISON.

2548. *A test for divisibility by* 19.

The following test (discovered by Mr. J. Kashangaki, a student of Makerere College, the University College of East Africa) does not appear to have been given before, and seems simpler to apply in practice than those given in Notes 1895 and 1986 (*Gazette*, Vol. XXXI, p. 239).

Cross off the last digit and add to the number remaining twice the digit crossed off. If this is divisible by 19, so is the original number. By successive application we reduce to a two-figure number (ultimately, if we wish, to a number not greater than 19).

E.g. to test 3086379 we obtain in succession:

$$\begin{aligned} 308637+2.9&=308655,\\ &\quad 30875,\\ &\quad 3097,\\ &\quad 323,\\ &\quad 38. \end{aligned}$$

Since 38 is a multiple of 19, so is 3086379.

The proof of this is very simple. F. M. ARSCOTT.

2549. *A graphical representation of quadratic equations.*

The following graphical representation of quadratic equations, which gives immediate information about the nature of the roots and can yield rough

approximations to them, was worked out experimentally by one of my pupils, Anthony Bayes, who is just taking his G.C.E. at ordinary level. I have merely supplied the proof. The method was certainly new to me, and may perhaps be unknown to others.

Let the equation be

$$x^2 + ax + b = 0. \quad \ldots\ldots(1)$$

Take rectangular cartesian coordinates plotting a against b so that every point in the plane represents an equation with real coefficients. The locus of all equations having equal roots is then the parabola

$$a^2 = 4b. \quad \ldots\ldots(2)$$

All points inside the parabola represent equations with complex roots and all points outside equations with real roots. Moreover, all equations having one real root in common lie on a tangent to this parabola and the common root is given by $\sqrt{}$(ordinate of the point of contact).

The condition that (1) should have one root α is

$$a\alpha + b + \alpha^2 = 0, \quad \ldots\ldots(3)$$

showing that the locus of equations with one common root is a straight line.

From (2) and (3), $\quad a^2 + 4a\alpha + 4\alpha^2 = 0,$

or $\quad (a + 2\alpha)^2 = 0\,;$

$$\therefore\ a = -2\alpha.$$

Thus (3) is a tangent to (2) at the point $(-2\alpha, \alpha^2)$. H. GEBERT.

2550. *On Note* 2355 *: a dielectric cylinder.*

The elegant results

$$\Omega_i(z) = \frac{2}{1+k} f(z), \quad \Omega_0(z) = f(z) + \frac{1-k}{1+k} \bar{f}\left(\frac{a^2}{z}\right), \quad \ldots\ldots(1)$$

for the potentials inside and outside a dielectric cylinder $|z| = a$, placed in a two-dimensional electrostatic field whose complex potential $f(z)$ in air has no singularities inside or on $|z| = a$, suggest that a solution entirely in terms of complex potentials should be possible without resort to a real variable proof.

1. The boundary conditions at $z\bar{z} = a^2$ may be written

$$\Omega_i(z) + \bar{\Omega}_i\left(\frac{a^2}{z}\right) = \Omega_0(z) + \bar{\Omega}_0\left(\frac{a^2}{z}\right),$$

$$k\left\{\Omega_i(z) - \bar{\Omega}_i\left(\frac{a^2}{z}\right)\right\} = \Omega_0(z) - \bar{\Omega}_0\left(\frac{a^2}{z}\right),$$

so that, when $z\bar{z} = a^2$,

$$2k\Omega_i(z) = (1+k)\,\Omega_0(z) - (1-k)\,\bar{\Omega}_0\left(\frac{a^2}{z}\right). \quad \ldots\ldots(2)$$

Now $\Omega_0(z)$ must contain a term $f(z)$ and $\Omega_i(z)$ must be regular in $|z| < a$, hence, from (2), it is seen that the complete form for $\Omega_0(z)$ must be that given by (1), since $\bar{f}(a^2/z)$ has singularities only in $|z| < a$. The remaining formula in (1) for $\Omega_i(z)$ then follows immediately from (2). It must, of course, be remembered that there is a uniqueness theorem in the background.

2. It may, perhaps, be of interest to add an alternative solution, including a proof of uniqueness, by a method due to Muskhelishvili.

Let $$\Omega_0(z)=f(z)+\omega(z) \quad (|z|>a), \qquad \ldots\ldots\ldots\ldots(3)$$

where $f(z)$ is regular in $|z|<a$ and $\omega(z)$ is regular in $|z|>a$ and $O(1/z)$ for large $|z|$. Without loss of generality $f(0)$ may be taken to be zero. Since $\omega(z)$ is defined in $|z|>a$, $\overline{\omega}(a^2/z)=\overline{\omega(a^2/\bar{z})}$ is defined in $|z|<a$. Now define

$$\omega(z)=\frac{2k}{1+k}\Omega_i(z)+\frac{1-k}{1+k}\overline{\omega}\left(\frac{a^2}{z}\right) \quad (|z|<a). \qquad \ldots\ldots\ldots\ldots(4)$$

Then the boundary condition (2) becomes

$$\omega^+(t)-\omega^-(t)=\frac{1-k}{1+k}\bar{f}\left(\frac{a^2}{t}\right)-f(t) \quad (|t|=a), \qquad \ldots\ldots\ldots\ldots(5)$$

where $\omega^+(t)$, $\omega^-(t)$ denote the limiting values of $\omega(z)$ as z tends to t, from the outside and inside of the circle $|z|=a$, respectively. Thus $\omega(z)$ is a sectionally holomorphic function in the entire z-plane, of order $1/z$ for large $|z|$, and with discontinuity on the circle $|z|=a$ given by (5). The solution can either be written down in the integral form

$$\omega(z)=\frac{1}{2\pi i}\oint\left\{f(t)-\frac{1-k}{1+k}\bar{f}\left(\frac{a^2}{t}\right)\right\}\frac{dt}{t-z},$$

the integral being anti-clockwise around the circle $|t|=a$, and then evaluated by residue theory, or it can be obtained by inspection from (5) in the form

$$\left.\begin{aligned}\omega(z)&=f(z) && (|z|<a),\\ \omega(z)&=\frac{1-k}{1+k}\bar{f}\left(\frac{a^2}{z}\right) && (|z|>a),\end{aligned}\right\}\qquad \ldots\ldots\ldots\ldots(6)$$

since $f(0)=0$. The difference of two possible solutions, each satisfying the conditions of the problem, is then holomorphic in the entire plane and $O(1/z)$ for large $|z|$, and is therefore identically zero, so that (6) is unique.

The formula (1) for $\Omega_0(z)$ follows from (3) and (6). Also, from (4) and (6),

$$\frac{2k}{1+k}\Omega_i(z)=f(z)-\left(\frac{1-k}{1+k}\right)^2 f(z),$$

which agrees with the first formula in (1). A. E. Green.

2551. *On nomograms.*

1. The most general relation, linear in each of three variables, may be written :

$$axyz+b_1yz+b_2zx+b_3xy+c_1x+c_2y+c_3z+d=0.$$

Let $$K=(b_1c_1+b_2c_2+b_3c_3-ad)^2$$
$$+4(ac_1c_2c_3+b_1b_2b_3d-b_1b_2c_1c_2-b_2b_3c_2c_3-b_3b_1c_3c_1).$$

Then if $K>0$ nomograms can be constructed in which :

the variables are separately represented on three non-concurrent straight lines, graduated with scales which are either linear or projections of linear scales,

the three variables are separately represented on three parallel, or concurrent, straight lines graduated with scales which are logarithmic functions of the variables,

two of the variables are represented on one circle and the third on an intersecting straight line,

all three variables are represented on one nodal cubic curve ;

and if $K=0$, nomograms can be constructed in which :

the variables are separately represented on three concurrent or parallel, straight lines, graduated with scales which are either linear or projections of linear scales,

two variables are represented on one circle and the third on a tangent line,

all three variables are represented on one cusped cubic,*

and if $K<0$, nomograms can be constructed in which :

the variables are separately represented on three parallel, or concurrent, straight lines whose scales are graduated in terms of arctangents of functions of the variables,

two variables are represented on one circle and the third on a non-intersecting straight line,

all three variables are represented on one cubic with an isolated singular point.

2. The most general relation, linear in two variables and of second degree in a third may be written

$$z^2(a_1xy+b_1x+c_1y+d_1)+z(a_2xy+b_2x+c_2y+d_2)+(a_3xy+b_3x+c_3y+d_3)=0.$$

It is assumed that the relation cannot be made linear in a function of z.

Then if

$$\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix} \cdot \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \cdot \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

nomograms can be constructed in which the variables x, y are represented on two straight lines, graduated with either linear scales or projections of linear scales, and the third variable, z, is represented on a circle.

If the above condition is not fulfilled, nomograms can be constructed in which x and y are represented on one circle (or a conic) and z is represented on a conic (or a circle). It is not always possible for both loci to be circles.

3. The most general relation between three variables which is linear in two of the variables may be written

$$f_1(z)xy+f_2(z)x+f_3(z)y+f_4(z)=0.$$

It is assumed that the relation cannot be made linear in a function of z.

Nomograms can be constructed if either, one of the four functions of z is identically zero, or numbers p, q, r, s, not all zero, can be found such that

$$p\,.\,f_1(z)+q\,.\,f_2(z)+r\,.\,f_3(z)+s\,.\,f_4(z)=0, \quad \text{all } z.$$

If one of the functions of z is identically zero, or if $ps=qr$, x, y, are represented on two straight lines and z on a curve depending on the nature of the functions. Otherwise x, y, are represented on a circle (or a conic) and z on a curve.

Double-sided scales are used to represent two variables on one curve, but the representation of three variables on one cubic presents difficulty. An interesting point about arctangent scales is that the whole range of values of the variables from $+\infty$ to $-\infty$ can be shown on a finite length, but more than one cycle of values may have to be drawn for one of the variables in order to accommodate all possible combinations of values of the other two.

C. V. GREGG.

2552. *The golden section and* π.

A line is divided in golden section if the whole line bears to the greater

* See Richard K. Guy, Note 2315, *Math.Gazette*, Feb. 1953.

part the same ratio that the greater part bears to the lesser part. The use of the golden section in architecture and the arts can be traced back to the Greeks, it is apparent in the work of such Italian painters as Giorgione, Piero della Francesca, and Titian, it was used extensively by Seurat last century, it is at the basis of the architecture of Corbusier, and it has many adherents amongst modern painters. The golden section ratio is also one of the fundamental proportions of organic growth. Thus it provides a common ground for discussion between mathematicians, artists, architects, and biologists.

The purpose of this note is to show that a simple construction to give the golden section of a line provides also a method of approximately squaring a circle or an ellipse.

Let AOB be a diameter and OD a radius at right angles to it in the circle ABD. Bisect OB at C and with radius CD draw an arc to cut AO in golden section at F. (This is a common construction for finding the golden section of a line AO).* Draw PFQ at right angles to AO cutting the circumference of the circle at P and Q. Join OP.

Then PQ is approximately equal to the arc AD, and the rectangle contained by PQ and AB is approximately equal to the area of the circle. For, calling the golden section ratio $\phi = \frac{1}{2}(1+\sqrt{5})$, and r the radius of the circle, we have, since $\phi^2 = \phi + 1$, and $\phi - 1 = \phi^{-1}$,

$$PF^2 = r^2 - r^2/\phi^2 = r^2/\phi = r^2(\phi - 1).$$

Hence

$$PQ = 2PF = 2r\sqrt{(\phi - 1)} = 1{\cdot}5723r \simeq \tfrac{1}{2}\pi r = \text{arc } AD,$$

and

$$PQ \,.\, AB \simeq \tfrac{1}{2}\pi r \,.\, 2r = \pi r^2.$$

The error in each case is less than 0·1%, since $\sqrt{(\phi - 1)} = 0{\cdot}78615$ and $\frac{1}{4}\pi = 0{\cdot}78540$.

A further but less satisfactory approximation may be noted. The length of the straight line joining A and P is approximately the length of the side of a regular heptagon inscribed in the circle. We have $AP = r(\phi - 1)\sqrt{2} = 0{\cdot}8740r$, while the side of the heptagon is $0{\cdot}8678r$. (Cf. Note 2297. The approximation in the present case is much inferior.)

The sides of the triangle OPF are in geometrical progression and show the relation

$$OP : FP = FP : OF = \sqrt{\phi}.$$

We have also

$$\angle FOP = \tan^{-1}\sqrt{\phi} = \operatorname{cosec}^{-1}\sqrt{\phi} = \sec^{-1}\phi = 51^\circ\,50'.$$

This is almost exactly the angle 51° 52′ of the Great Pyramid.†

To square an ellipse, though not to rectify an arc, the outer golden section F of a semi-axis OA $(=a)$ is found and a line is drawn through it at right angles to cut the circumference of the ellipse at P and Q. We then have

$$2a \,.\, PQ \simeq \text{area of ellipse.}$$

For, if b be the other semi-axis,

$$PF^2/b^2 = 1 - OF^2/a^2 = 1 - (\phi - 1)^2 = \phi - 1.$$

Thus $FP = b\sqrt{(\phi - 1)}$, and

$$2a \,.\, PQ = 4ab\sqrt{(\phi - 1)} \simeq \pi ab.$$

G. S. Smith.

* M. Ghyka, *Geometrical Composition and Design* (1952).

† M. Ghyka, *Geometry of Art and Life* (1946); I. E. S. Edwards, *Pyramids of Egypt*, 87 (1952).

2553. *Extraction of square roots.*

At the end of his article on the extraction of square roots (*Gazette*, XXXV (1951), pp. 89–90) Professor Haldane writes : " The methods here given are probably now of no practical importance. Had they been discovered in the 17th century, as they might have been, they would have saved a good deal of computation."

In point of fact Brook Taylor in 1717 gave " A general Series for expressing the Root of any Quadratick Equation ". It will be found towards the end of his paper " An attempt towards the Improvement of the Method of approximating, in the Extraction of the Roots of Equations in Numbers ", *Phil. Trans. Royal Soc.*, XXX (1717–19), pp. 610–22.

In modern notation Taylor's solution of the quadratic equation

$$xx - akx + akk = 0$$

is

$$x = k + \frac{k}{c} + \frac{k}{cc'} + \frac{k}{cc'c''} + \dots ,$$

where $c = a - 2$, $c' = c^2 - 2$, $c'' = c'^2 - 2$, ... He gives the example

$$1 + \sqrt{2} = \frac{1}{2} - \frac{1}{2 \cdot 6} - \frac{1}{2 \cdot 6 \cdot 34} - \frac{1}{2 \cdot 6 \cdot 34 \cdot 1154} - \frac{1}{2 \cdot 6 \cdot 34 \cdot 1154 \cdot 1331714} - \dots .$$

and concludes " The Fractions here wrote down giving the Root true to twenty three Places ".

A. OPPENHEIM.

2554. *The inflexions of a real non-singular cubic.*

The real non-singular cubic has nine inflexions, of which precisely three are real. The following proof, though unlikely to be original, may be of interest, as it does not involve either considerations of the roots of a cubic or considerations of circuits as in Hilton's book.

It is well known that the nine inflexions lie by threes on twelve lines, four through each inflexion. We may denote the collinear triads by the scheme

123	189	279	357
145	246	368	478
167	258	349	569

Suppose first that more than three are real. Suppose in fact that 1 and 2 are real ; 3 must then be real. We can also suppose 4 real, and it is then easy to see that all are real, from the collinearities.

Of course, in any case one inflexion is real ; let us in fact write $A = 1$ (real) and assume now that there is a complex line of inflexions through A, say 123, and write $P = 2$, $Q = 3$, where $Q \neq \overline{P}$. The complex conjugate line to 123 is another line of inflexions through A ; we may suppose it 145 and write $\overline{P} = 4$, $\overline{Q} = 5$. Then $\overline{P}Q = 349$ is the complex conjugate line of $P\overline{Q} = 258$. These two lines do *not* meet in an inflexion and so 8 and 9 must be complex conjugate ; write $R = 8$, $\overline{R} = 9$. QR now meets $\overline{Q}\overline{R}$ at 6, which is then a *real* inflexion B. AB then meets the cubic again in a real inflexion $C = 7$.

We thus see that there are three mutually exclusive possibilities :

(i) all inflexions are real ;
(ii) there are precisely three real (collinear) inflexions ;
(iii) there is a real inflexion A, and all the four lines joining A to the other eight inflexions are real.

We can finally discard possibilities (i) and (iii) by showing that the four lines-of-inflexions through an inflexion form an equiharmonic pencil, and so are not all real. To see this, let 246 and 189 meet at O. We then have

$$1(2460) = 5(2460) = (8190),$$

but also $$1(2460) = 3(2460) = (1980).$$

But of course, $$(8190) = \{(1980) - 1\}/(1980)$$

and is equiharmonic. We are thus left with possibility (ii). C. F. FISHER.

2555. *Polygons inscribed in polygons.*

In the December, 1952, *Gazette*, Mr. H. V. Lowry showed that it was possible to inscribe an infinity of hexagons in any triangle, having the three pairs of opposite sides parallel to the three sides of the triangle. He then extended the property to other types of hexagons, and also to polygons. In fact, general properties are given very neatly for the triangle by non-metrical methods.

THEOREM A. *ABC is any triangle, P_1 a variable point on BC, and X, Y, Z any three fixed points.*

Z projects P_1 on BC into P_2 on CA ;
X projects P_2 on CA into P_3 on AB ;
Y projects P_3 on AB into Q_1 on BC ;

then P_1Q_1 is a point-pair of a homography on BC.

This follows immediately from the properties of ranges in perspective.

THEOREM B. *If CA meets BZ, BX at M, M', and MX meets AB at V, $M'Z$ meets BC at V', then in Theorem A, the homography on BC is an involution if and only if Y lies on VV'.*

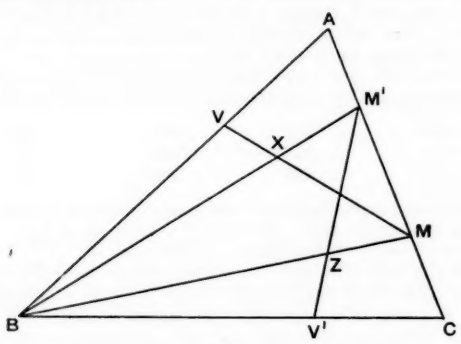


FIG.

For : Z projects V' on BC into M' on CA ;
X projects M' on CA into B on AB ;
Y projects B on AB into B on BC, wherever Y may be.

Hence V', B is a point pair of the homography on BC.

But now, starting with B,

Z projects B on BC into M on CA ;
X projects M on CA into V on AB ;
Y projects V on AB into V' on BC, if and only if Y lies on VV'.

That is, B, V' is also a point pair of the homography if and only if Y lies on VV'.

But a homography with a symmetrical point-pair is an involution. Thus the homography is an involution if and only if Y lies on VV'.

If the homography is an involution, every point-pair is symmetric. Hence if $P_1 \to Q_1$ on BC, $Q_1 \to P_1$, and the hexagon is closed for every position of P_1.

It is clear that the triangle XYZ stands in a special relationship to the triangle ABC, and that the relationship can be expressed equally well in terms of any of the three vertices X, Y, Z. If we call XYZ a locus involution triangle of the triangle ABC, we may then prove quite easily that ABC is a dual, or tangential involution, triangle of the triangle XYZ.

Returning to the original problem, we have found the necessary and sufficient condition for a hexagon to exist inscribed in a triangle, having its opposite pairs of sides meeting at three given points X, Y, Z. Also from the foregoing theory, if one such hexagon exists, there is an infinite number of them.

THEOREM C. *If X, Y, Z lie on the three sides of the triangle ABC, then XYZ is an involution triangle of the triangle ABC.*

For, referring to theorem B, M and M' lie at A and C. Therefore V and V' lie at A and C. Hence Y lies on VV'. Thus XYZ is an involution triangle of ABC.

It follows that there is an infinite number of hexagons inscribed in the triangle ABC, whose opposite sides meet at X, Y, Z. It is interesting to see how this last observation arises from the preceding theory; but in fact it is merely the dual of Pappus in disguise.

Taking X, Y, Z at infinity on the sides of the triangle ABC, we get Mr. Lowry's original inscribed hexagon having the three pairs of opposite sides parallel to the three sides of ABC. S. N. COLLINGS.

2556. *A proof of the cosine rule.*

In any triangle ABC,

$$a = b\cos C + c\cos B,$$

with similar formulae for b and c.

$$\begin{aligned} b^2 + c^2 - 2bc\cos A &= b(b - c\cos A) + c(c - b\cos A) \\ &= ba\cos C + ca\cos B \\ &= a(b\cos C + c\cos B) \\ &= a^2. \end{aligned}$$

P. GANT.

2557. *Notation for the logarithm of a negative number.*

In a certain piece of logarithmic computation, the need for a device to indicate the logarithm of a negative number was experienced, and I found that the notation to be described saved oversights and waste of time.

Let us adopt the convention that a point written below or above the decimal point between the characteristic and the mantissa of a logarithm denotes the addition to that logarithm of the imaginary logarithm of -1. Thus if (as in English textbooks) $2{\cdot}0000 = \log 100$, or (as in American books) $2.0000 = \log 100$, or (as in French books) $2{,}0000 = \log 100$, then $2{:}0000$ (in the English or American notation) or $2;0000$ (in the French) will represent

$$\log 100 + \log(-1) = \log(-100).$$

Being typographically identical with a colon (or, for the French notation, with a semi-colon) this device does not (as do most new symbols needed with

development of mathematical technique) need the cost or inconvenience of a new symbol in printing founts or on typewriters; and it does not add (as even a plus or minus does) to the widths of columns on a possibly overcrowded working sheet used by a computer; but it does give the latter a conspicuous visual reminder.

Clearly the rule of signs affects addition of logarithms thus distinguished: for example, the sum of an even number of double-point logarithms is a single-point logarithm, and the sum of an odd number of them is a double-point logarithm. Thus

$$1{:}0000 + 2{\cdot}0000 + 3{\cdot}0000 = 6{:}0000,$$

but

$$1{:}0000 + 2{:}0000 + 3{\cdot}0000 = 6{\cdot}0000.$$

COLIN EVANS.

2558. *Angular velocity of a rigid body.*

The derivation of the expression given, for example, in Milne's *Vectorial Mechanics*, 198, for the angular velocity of a rigid body assumes the form of the equation $\dot{\mathbf{r}} = \boldsymbol{\Omega} \wedge \mathbf{r}$ for the velocity of a point of the body. A direct approach is, however, possible.

Let $\mathbf{r}$, $\mathbf{r}_1$, $\mathbf{r}_2$ be the position vectors relative to a fixed point O of three points of a rigid body rotating about O, and let $\dot{\mathbf{r}}$, $\dot{\mathbf{r}}_1$, $\dot{\mathbf{r}}_2$ be their velocities. Then the conditions of rigidity are

$$\mathbf{r}_i^2 = \text{constant}, \quad (\mathbf{r}_i - \mathbf{r}_j)^2 = \text{constant}.$$

Differentiating with respect to time, we have

$$\dot{\mathbf{r}}\mathbf{r} = 0, \quad \dot{\mathbf{r}}\mathbf{r}_1 = -\mathbf{r}\dot{\mathbf{r}}_1, \quad \dot{\mathbf{r}}\mathbf{r}_2 = -\dot{\mathbf{r}}_2\mathbf{r}.$$

If $\mathbf{r}$, $\mathbf{r}_1$, $\mathbf{r}_2$ are not coplanar, we can write $\dot{\mathbf{r}}$ in the form

$$\dot{\mathbf{r}} = \lambda(\mathbf{r}_1 \wedge \mathbf{r}_2) + \mu(\mathbf{r}_2 \wedge \mathbf{r}) + \nu(\mathbf{r} \wedge \mathbf{r}_1),$$

where $\dot{\mathbf{r}}\mathbf{r} = \lambda(\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2)$, and two similar expressions give the values of μ, ν.

Thus

$$\begin{aligned}(\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2)\dot{\mathbf{r}} &= -(\mathbf{r}\dot{\mathbf{r}}_1)(\mathbf{r}_2 \wedge \mathbf{r}) - (\dot{\mathbf{r}}_2\mathbf{r})(\mathbf{r} \wedge \mathbf{r}_1) \\ &= \{(\dot{\mathbf{r}}_2\mathbf{r})\mathbf{r}_1 - (\mathbf{r}\dot{\mathbf{r}}_1)\dot{\mathbf{r}}_2\} \wedge \mathbf{r} \\ &= \{(\dot{\mathbf{r}}_2\mathbf{r})\mathbf{r}_1 + \dot{\mathbf{r}}_1 \wedge (\mathbf{r} \wedge \mathbf{r}_2)\} \wedge \mathbf{r}.\end{aligned}$$

Now, since $\dot{\mathbf{r}}_1\mathbf{r}_1 = 0$,

$$\begin{aligned}-(\mathbf{r}_1\dot{\mathbf{r}}_2)\dot{\mathbf{r}}_1 \wedge (\mathbf{r} \wedge \mathbf{r}_2) &= \{(\dot{\mathbf{r}}_1 \wedge \dot{\mathbf{r}}_2) \wedge \mathbf{r}\} \wedge (\mathbf{r} \wedge \mathbf{r}_2) \\ &= -(\dot{\mathbf{r}}_1 \wedge \dot{\mathbf{r}}_2)(\mathbf{r}_1 \,.\, \mathbf{r} \wedge \mathbf{r}_2) + \mathbf{r}_1\{(\dot{\mathbf{r}}_1 \wedge \dot{\mathbf{r}}_2) \,.\, (\mathbf{r} \wedge \mathbf{r}_2)\} \\ &= (\dot{\mathbf{r}} \wedge \dot{\mathbf{r}}_2)(\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2) + \mathbf{r}_1\{(\dot{\mathbf{r}}_1\mathbf{r})(\dot{\mathbf{r}}_2\mathbf{r}_2) - (\dot{\mathbf{r}}_1\mathbf{r}_2)(\dot{\mathbf{r}}_2\mathbf{r})\}\end{aligned}$$

or

$$\dot{\mathbf{r}}_1 \wedge (\mathbf{r} \wedge \mathbf{r}_2) = -(\dot{\mathbf{r}}_1 \wedge \dot{\mathbf{r}}_2)\left(\frac{\dot{\mathbf{r}}, \mathbf{r}_1, \mathbf{r}_2}{\mathbf{r}_1\dot{\mathbf{r}}_2}\right) - \mathbf{r}_1(\dot{\mathbf{r}}_2\mathbf{r}).$$

It follows that

$$\dot{\mathbf{r}} = \left(\frac{\dot{\mathbf{r}}_1 \wedge \dot{\mathbf{r}}_2}{\dot{\mathbf{r}}_1\mathbf{r}_2}\right) \wedge \mathbf{r}.$$

Thus the angular velocity $\boldsymbol{\Omega}$ is given in terms of the motions of any two particles of the body by the expression

$$\boldsymbol{\Omega} = (\dot{\mathbf{r}}_1 \wedge \dot{\mathbf{r}}_2)/(\dot{\mathbf{r}}_1\mathbf{r}_2).$$

R. B. HARVEY.

6. Isotropic lines.

Join any four collinear points A, B, C, D to any point P not collinear with them, and project PB and PD into circular lines. Denote the respective projections of P, A, B, C, D by P', A', B', C', D'. Then

$$\{ABCD\} = P\{ABCD\} = P'\{A'B'C'D'\}$$
$$= \frac{\sin A'P'B' \sin C'P'D'}{\sin A'P'D' \sin C'P'B'}.$$

But $\sin A'P'B' = \sin C'P'B'$, and $\sin A'P'D' = \sin C'P'D'$, since $P'B'$ and $P'D'$ are isotropic lines. Hence

$$\{ABCD\} = 1.$$

[The ratio of the sines is indeterminate, since $\sin(\tan^{-1} i)$ is infinite. It is meaningless to say that an isotropic line makes the same angle with all lines, as the imaginary part of $\tan^{-1} i$ is infinite. An example of this fallacy occurs in Askwith's *Pure geometry*, § 130.] A. A. G.

7. The four points of intersection of two conics are collinear.

LEMMA. *Let two given conics meet in four distinct points A, B, C, D, and let a pair of common tangents touch the first at P, Q and the second at P', Q'; let PQ meet $P'Q'$ in R. Then the line RD passes through a further point of intersection of the two conics.*

This problem was set in a recent examination paper at Scholarship level. The proof which follows was the one which appeared most often in the solutions.

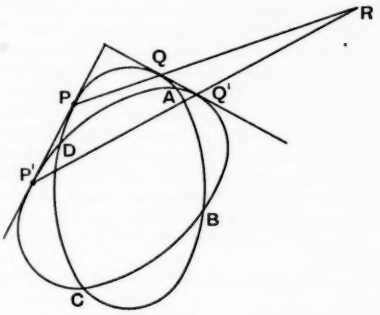


FIG. 1.

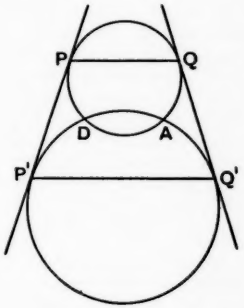


FIG 2.

Project B, C into the circular points at infinity. We obtain the figure shown in Fig. 2, with the conics now as circles. Clearly DA is parallel to each of PQ, $P'Q'$ (themselves clearly parallel) and so, in the original figure (Fig. 1), DA passes through R.

We are now able to prove our main theorem, which seems to have escaped the attention of candidates. We have proved, by projecting B, C, that A lies on RD; similarly, by projecting C, A, we may prove that B lies on RD; and, by projecting A, B, that C lies on RD.

Hence A, B, C, D are collinear. E. A. MAXWELL.

REVIEWS

Elementary Mathematics. By C. G. Nobbs. Part 1. Pp. 336. 9s. 6d., Part 2. Pp. 334. 9s. 6d. 1954 (Oxford University Press).

These books for use in secondary schools are the first of a series of four which take the subject up to the O-level of the G.C.E., including Coordinate Geometry and Calculus, the course having been written with the Alternative Syllabuses particularly in mind and having arrived at its present shape after extensive trial in the classroom.

There is an intentional overlap between successive parts, and most topics are treated twice over, often from different points of view. For example, in Part 1 the theorem of Pythagoras is proved by means of dissection whereas in Part 2 it appears first as an exercise in calculation and is later proved by Trigonometry. For those who have time to cover the whole of each book, revision thus becomes a live and integral part of the work, while for slower pupils it is possible to make a fresh start on a new book without having finished the previous one, though in this case the teacher may need to supplement the bookwork.

With regard to the amount of bookwork included, the author steers a middle course and achieves his intention of leaving teachers free to use their own methods of presentation while at the same time including enough to be of use to a pupil after a period of absence. The beginnings of Geometry and Algebra, and the introductions to some other topics, will certainly require amplifying by the teacher, but on the other hand there are plenty of worked examples carefully set out for the guidance of the pupil. The point of view of both teacher and pupil is respected, and in the setting out of the solution of simple equations and elsewhere a nice compromise is effected between the teacher's desire for explanations and the pupil's dislike of giving them. The history of the relevant parts of mathematics has been worked into the text and should prove informative and stimulating, while the exercises are numerous, well-graded, and the choice of material for them is unusually wide.

Part 1 consists of 132 sections and 149 exercises, Part 2 of 152 sections and 138 exercises. The books are not divided into chapters, and the number of an exercise usually differs from that of the section to which it applies, but nearly all sections have a title and both section and exercise numbers are clearly shown at the head of each double page ; it is thus reasonably easy to find one's way about, particularly since each book in fact consists of sequences of Arithmetic, Algebra, Geometry, and later Trigonometry, with a bridge from one subject to the next where possible.

Within the short space of a review the contents cannot be described in any detail, but a brief indication follows.

Part 1. In the first 112 sections Arithmetic is revised from the beginning and taken as far as ratio, percentage and graphs ; the notions of Geometry are introduced and developed, reaching congruence and the idea of geometrical proof ; the introduction to Algebra is carefully carried out and the treatment of the subject includes brackets, fractions, equations and problems. The last 20 sections move rather fast and seem intended for the brighter pupils. They are built round the idea of a square and include squares and higher powers in Algebra, Pythagoras' theorem, the graph of squares, and the mensuration of the circle and cylinder.

A section on Estimates is excellent, but the treatment of Ratio and Proportion seems to under-estimate the difficulties of the subject and the need for clarity. For example, ratios are identified with fractions (" the ratio $\frac{2}{3}$ ") and this treatment is immediately followed by ratios of type 2 : 5 : 7. One omission

should be mentioned : the reviewer can find no reference to the multiplication or division of compound quantities.

Part 2. In the first 141 sections logarithms are carefully and thoroughly introduced and applied, using 3-figure tables ; Geometry is taken as far as the circle theorems, Algebra to simultaneous equations ; Trigonometry is developed from the idea of similarity, the three principal ratios being brought in and applied. The last 11 sections deal with products and factors in Algebra, leading to quadratic equations, but no knowledge of them will be presumed in Part 3.

The treatment throughout Part 2 is admirable and the only criticism worth making is that, if the work is taught in the printed order, the pupils will at one stage find themselves spending a whole term doing nothing but Arithmetic.

The printers have done their work well : the only error which has come to the reviewer's notice is the omission of the page-number of § 1 in the table of contents to Part 1.

Both books have the workmanlike qualities which would be expected of the author of *Elementary Calculus and Coordinate Geometry*, and those who like to use only one mathematical textbook at a time will find in this series a valuable addition to the range at their disposal and will look forward with interest to the appearance of Parts 3 and 4. A. B.

A History of Science. Ancient Science through the Golden Age of Greece. By GEORGE SARTON. Pp. xxvi, 646. 63s. 1953. (Harvard University Press. London, Geoffrey Cumberlege.)

This fine volume is the first of a series of eight, the whole being based on four courses of thirty-five lectures each. It should be said at once that the size of the projected work should not cause anyone to hesitate before embarking on this first volume ; for it is complete in itself and well worth reading even by those who do not intend to go further. It does indeed set a standard which shorter histories cannot hope to live up to, and the reader may thus be in danger of acquiring expensive tastes. This is not to say that a scholar would not ask for fuller treatment in many sections : in particular the chapter called " Assyrian Intermezzo " is disappointingly short, and the two pages devoted to the Book of Job raise more questions than they answer. But, by and large, Dr. Sarton has tackled successfully the difficult problem of selection. He had already found that in a shorter work he could not re-create what was evidently the interesting and vivid style of his lectures. He says in the Preface that, as he grew older, his lectures became simpler ; he tried to say fewer things and to say them with more humanity.

One-fifth of the present volume is concerned with pre-Greek science, mainly in Egypt, Mesopotamia and the Aegean area. It does not deal with Hindu or Chinese science because, as Dr. Sarton says, that is not part of our ancestry. He is much concerned with the influence on the Greeks of their immediate predecessors in the Middle East. Only a work on this scale can describe the position fairly. We are too accustomed to short summaries which minimise the pre-Greek achievements and we have read so much by admirers of the Greeks about the rational nature of Greek thought contrasted with the empirical science of Egypt and Babylon. " Homer and Hesiod did not come out of nothing." He protests too against the confusion made by historians who deal with Mesopotamian science as if it were a single entity prior to Greek science, and he distinguishes three periods : the golden age of Babylonian science in the second millenium B.C., the " Assyrian " science of the seventh century, and the " Chaldean" science of Hellenistic times. The last period is not dealt with in this volume, and in fact most attention is given to the first. We learn, for example, that the Sumerians had the idea of position

value, though without a zero, and even extended it to sub-multiples of the base. They had tables of reciprocals. The Greeks discarded all this and mixed up the sexagesimal and decimal systems; they retained the worst features of Sumerian arithmetic and overlooked the best.

The Greek chapters, which form the bulk of the book, are absorbing reading. It is difficult to imagine a better way for the scientist to approach the study of Greece. As in earlier chapters, we learn a great deal about the ancient world apart from its scientific achievements; which is natural enough, because science did not develop *in vacuo*, but was conditioned by its environment and in turn influenced the world outside itself. Of course there is among original thinkers a certain detachment from the world, and the author recognises this, but even in the realm of abstract thought there is a "climate" making some things possible and forbidding others. This is shown by the fact that discoveries have so often been made simultaneously by two or more thinkers and also by the fact that discoveries made before their time have been forgotten and lost.

The Greeks had many ideas that were, in this sense, premature, and they asked many questions that could not be answered, some indeed that have not been answered yet. They were often groping blindly and they sometimes developed fanciful ideas on insufficient knowledge. Yet they showed many examples of true scientific spirit and method. How remarkable was the work of Eudoxus on proportion; and how rigorously was his "method of exhaustion" presented. How admirably objective was Hippocrates' account of a patient's sickness, or, for that matter, Thucydides' account of the Peloponnesian war. And how masterly was Aristotle's synthesis of the knowledge of his time. Thucydides is described as the first scientific historian, Herodotus as the first ethnologist; Plato the first sociologist, and his Academy the first institution for higher education. But Plato was also the first theorist of totalitarianism and it is a far cry from his Republic to the ideas of Western democracy.

Dr. Sarton is in fact strongly critical of Plato, who witnessed the downfall of Athens and admired Sparta; who proposed a Utopia in which nothing would ever change and eighty per cent of the population would be slaves; whose "ideal of frozen perfection entailed a hatred not only of freedom but of individualism in all its forms"; whose lack of humanity was a "betrayal" of his master Socrates. Yet "Plato was a poet and a metaphysician, a craftsman who used wonderfully well a literary instrument of almost unbelievable exquisiteness, the Greek prose of the Golden Age." Dr. Sarton is in two minds whether we should treat Plato as Plato himself treats Homer—"crown him with flowers and drive him out of the city".

Is this an exaggerated and unfair view? Many will think so. The unfairness consists in a failure to recognise that Plato was describing a theory, not talking practical politics. His Republic was a paper model of a city and could not represent the change and development of a living community. Dr. Sarton condemns the scheme for not being logical and consistent when he means that it was not alive, not practical. The idea, for example, of a philosopher-king is eminently unpractical, but in the Republic a logical necessity. Dr Sarton is writing under the shadow of Hitler and Stalin, after some centuries in which the cult of Plato has become deeply ingrained and often been blindly followed, all of which makes it difficult to view the matter in perspective or to speak with moderation. So he calls Plato "a disgruntled fanatic" and the Republic "this crazy commonwealth".

Enough has been said to show that with Dr Sarton the history of science is very much alive. The book is packed with information, but is well-written, readable and stimulating. Anyone who has read it will be looking forward to the next volume.

E. H. L.

Tensor Analysis for Physicists. By J. A. SCHOUTEN. Second edition. Pp. xii, 277. 30s. 1954. (Geoffrey Cumberlege, Oxford University Press.)

Three years have brought a call for a second edition of this valuable work. This edition is revised, with some additional notes, but no extensive changes.

The name gives little idea of the exhaustive and exact treatment, including such things as a complete discussion of the various groups involved and the use of non-holonomic co-ordinate systems. Rather it is meant to imply the author's attempt (largely successful) to give the fruits of his long experience of lecturing in the form of physical pictures of the concepts used at each stage. The kernel index method makes the working very much simpler, so that the theory may be dealt with in 125 pages, and not many more are sufficient to discuss quite deeply such topics as elasticity (including, for instance, the quartz oscillator), classical dynamics, special and general relativity, and Dirac's quantum mechanics. The fifteen pages of summarised theoretical results are marked for easy reference. The book is excellently printed, and there is a bibliography of nearly 100 references, from Klein's Erlangen Program to the present day.

C. W. KILMISTER.

Calculus : A Modern Approach. By KARL MENGER. Second, enlarged edition. Illinois Institute of Technology, 1953. Pp. xxiv, 303.

This is no ordinary calculus text-book. For its author aims at nothing less then effecting a revolution in our conventional teaching of the calculus. At a first glance it appears that this is to be a revolution only in the notation in which the ideas of the calculus are expressed ; but in addition to these important notational changes there are no less important changes in the emphasis laid on the various parts of the calculus.

The cautious and sceptical reader, before examining a detailed account of the changes proposed by the author, will naturally expect to be given reasons for departing from the traditional presentation, which has served so many generations of mathematicians. The author's thesis, which he advances with force and good humour in his Introduction, is that the classical notations, while well enough adapted to mechanical techniques of manipulation, are consistent neither with themselves nor with each other, and conceal rather than reveal any understanding of the principles behind these techniques. In some cases these classical notations cloak with a false simplicity some of the deep results of the theory ; in particular they present as a truism the most important result of all, the relation between areas and antiderivatives. The author exposes also a great confusion about the meaning of the word " variable ", which he contends is used in the literature in two completely different senses. Even if his teaching experience and the author's arguments do not convince him of the justice of these charges against the traditional methods, the conservative reader ought not to lay down the book at this stage, for he may nevertheless find much in the author's treatment to interest him.

Perhaps the most satisfactory way to discuss the author's ideas is to outline an elementary calculus course based on his book, commenting on points of detail in the light of experience gained in teaching such a course.

The course begins with a statement of the two fundamental problems of the calculus, in the following terms. Let C be a simple curve (i.e. a curve which is cut once only by each perpendicular to the x-axis) ; the height of this curve above the point x on the x-axis is denoted by Cx, for the author has a dislike of parentheses, and here he regards them as unnecessary. The first fundamental problem is the determination of the slope of the tangent to C above the

point x, which is denoted by DCx. The other main problem is the calculation of the area bounded by the x-axis, the curve C and the ordinates at a and b; this area is denoted by $\int_a^b C$; compelling reasons are given for omitting the customary $x\,dx$. The reviewer wonders whether there is any point in retaining even the integral sign; the notation $A(C; a, b)$ seems more transparent.

These two fundamental problems are first solved in the simplest possible cases—the slope problem for non-vertical straight lines and polygons composed of non-vertical segments, and the area problem for horizontal straight lines and step lines. A natural impatience to reach " real " calculus ought not to dull the reader's appreciation of the sound pedagogical principle according to which the author includes these " calculus-free " discussions. For they enable him to introduce the basic ideas of slope line and area line, and to exhibit the striking relation between them uncomplicated by considerations of the concept of limit. Thus, even at this very early stage, the student painlessly becomes accustomed to thinking of differentiation and integration as operations which, when applied to a curve, yield respectively a single curve (the slope line) and a whole class of curves (the area lines).

The author next proceeds to show how a general simple curve can be approximated, on the one hand by polygons, and on the other by step lines, and how these approximations can be used to draw approximate slope lines and area lines respectively. There is again sound principle underlying the inclusion of this unconventional matter: it is so arranged that the formal definitions of derivative and integral, which are given later, appear quite natural.

Only now, when the student has been made familiar with the geometrical aspects of the problems, is he introduced to the ideas of function and limit. The author very properly defines the word " function " and not the time-hallowed phrase " y is a function of x ", which raises too many questions about the nature of y and x (no one who has read this book can ever again glibly describe them as " real variables "). A function, then, is defined as a class of pairs of real numbers in which two pairs which have the same first element have also the same second element; if (a, b) is one of the pairs in the class which constitutes the function f, one writes $b = fa$—again parentheses are unnecessary—and calls b the value of f at a. This is a far cry from the classical definition of function, but it is very close to the idea of a simple curve which has been before the student's eye since his course began; so to him, who has never seen the classical definition, the present definition is much less strange than it is to the more sophisticated reader. This definition has the further advantage that it makes it possible to distinguish clearly between the function f itself, which is a class of pairs of numbers, and the value of the function at x, fx, which is a number: who, asks the author, would confuse a party of married couples with a spinster? Failure to make this distinction bedevils the conventional treatments of the calculus, and is the principal cause of their worst confusion.

The necessity of distinguishing between a function and its values entails the introduction of some new symbols, notably the symbol I for the identity function, which consists of all the pairs (x, x), and which is conventionally called " the function x ". Indeed, in an earlier article, the author has declared that as long as we refrain from introducing a special symbol for the identity function we are comparable to virtuosi in multiplication without a symbol for the number 1.

Functions can be added and multiplied: the sum, $f+g$, and product, $f \cdot g$, of the functions f and g are the functions which pair with every number x the numbers $fx + gx$ and $fx \cdot gx$ respectively. Functions can also be combined

by the operation of substitution. The result of substituting the function g into the function f is the function which pairs with every number x the number $f(gx)$; this composed function is denoted by fg. The reviewer is rather unhappy about the very slight notational distinction between the product $f \cdot g$ and the composed function fg: students are notoriously inaccurate in matters of notation, and will think nothing of dropping the point in $f \cdot g$. A more definite distinction may be made by writing $f \circ g$ instead of fg for the composed function; this notation is already standard in many branches of mathematics, but its use would certainly entail the introduction of more parentheses than the author allows himself. By using the idea of substitution of functions the author gives a very clear and satisfactory account of inverse functions, a topic which the student usually finds very mysterious.

A number of examples is given to illustrate the idea of limits, followed by the formal definition: L is the limit of f at a if fx is arbitrarily close to L for all numbers x sufficiently close to a. To use the phrase "the limit of f at a" is in effect to return to the terminology of the nineteenth century, "the limit of $f(x)$ when $x=a$"; this terminology went out of fashion and was replaced by "the limit of $f(x)$ as x tends to a" because it was felt that it gave rise to confusion between the limit and the value of f at a. The reviewer feels that the "tends to" terminology has done little to resolve this confusion in the average student's mind; and it has had the very unfortunate effect of making him think of "x" as a bead sliding along the axis towards a—logically, of course, he has no right to think in this way, but psychologically the terminology is quite compelling.

It is now possible to discuss the solution of the two fundamental problems for simple curves in general. The derivative of the function f is defined (or proved, depending on one's level of sophistication) to be the function Df which pairs with each number a the number $Dfa = \lim_{a} \frac{f-fa}{I-a}$; the area under the graph of f between a and b, $\int_a^b f$, is defined (or proved) to be a limit of approximations by step lines. Immediately, without any discussion of the techniques of applying these definitions, the author proceeds to display the relationship between them. The a-area line of f is the function $\int_a f$ which pairs with each number b the area $\int_a^b f$; it is proved in the usual way that if F is any area line of f, then $DF=f$, i.e. F is an antiderivative of f. This result, however, goes only part way towards solving the problem of finding $\int_a^b f$; to complete the solution we require the Fundamental Theorem, that any two antiderivatives of f differ by a constant function. This, however, is a deep result, depending on the Mean Value Theorem, and no proof is offered at the present stage. Using this result, it follows immediately that $\int_a^b f = Fb - Fa$, where F is any antiderivative of f. It is good to have the Fundamental Theorem so clearly treated: too many text books either ignore the necessity of proving it, or else "prove" it by the dubious logical device of proving its converse. It is also good to have the central ideas of the calculus gathered together and clearly displayed in one chapter, uncomplicated by discussions of manipulative techniques; the Arts student who is interested more in ideas than in manipulation might well stop here.

For those who must learn to manipulate, the new notation is just as easy to operate as any of the classical ones. The rules for differentiating sums and products of functions look familiar enough; the "function of a function"

rule becomes $D(fg) = ((Df)g) \cdot Dg$. In practice, this rule is neither more nor less difficult to apply than the conventional " chain rule " $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ —it requires precisely the same dissection of the function into a chain of substitutions. Using the substitution rule it is possible to find the derivatives of inverse functions ; this method smacks less of illegality than the usual one. The author also gives a very satisfactory treatment of the differentiation of implicit functions, but unwisely relegates it to an exercise.

For the exposition of the calculus of antiderivatives, the author introduces the notation $D^{-1}f$ for the class of functions whose derivative is f ; unfortunately he sometimes slips away from this definition and uses $D^{-1}f$ to mean any member of this class of functions. All the standard techniques of antiderivation can be expressed in this notation without the use of variables : the reader will readily write down the sum rule, the constant factor rule, and the rule for integration by parts. Even the easier type of " integration by change of variable "—where the integrand is given in the form $fg \cdot Dg$—does not look too unfamiliar : $D^{-1}(fg \cdot Dg) = (D^{-1}f)g$. This rule, of course, follows at once from the " function of a function " rule for derivatives ; so does the second type of integration by change of variable—that usually expressed as " set $x = g(\theta)$; then $\int f(x)dx = \int f(g(\theta))\frac{dx}{d\theta}\,d\theta$ " ; in the new notation this becomes

$$D^{-1}f = (D^{-1}(fg \cdot Dg))g^*$$

where g^* is an inverse of g. No doubt this formula appears more complicated than the traditional one : but the process is in fact more intricate than the traditional one suggests—for the work is not finished when one finds

$$\int f(g(\theta))\frac{dx}{d\theta}\,d\theta :$$

this appears as a " function of θ ", which must then be " expressed in terms of x ". This last step corresponds, as the reader will see on a little reflection, to the substitution of the inverse function g^*.

The course on pure calculus concludes with an excellent discussion, fresh and clear, of the Mean Value Theorem, Taylor's Theorem and their applications, and a rather exhibitionistic chapter on " functions of two variables ", which is apparently included to convince the reader that the power and the advantages of the new notation are not confined to the elementary part of the calculus.

How now is this pure and rather severe mathematical discipline applied to physical problems? The chapter which answers this question is the longest in the book and the least clearly written, but it is worth the effort of reading, for it carries out a deep investigation, deeper perhaps than any previously attempted, of the meaning of the sentence " the distance travelled by a falling body is a function of the time " and of the equation " $s = 16t^2$ ". To express the idea as succinctly as possible, the reviewer is inclined to introduce the notion of mapping : the author is unwilling to do this, and lands himself in a welter of nomenclature talking of " pairings ", " numerical scales ", " variable quantities " and " functions ", all of which are simply special types of mappings. Let then Σ and T be two sets—either of real numbers or of " physical observables "—e.g. let Σ be the set of all space intervals traversed by a falling stone, and let T be the set of all time intervals during which the stone is falling. A mapping ϕ of T into Σ is a class of pairs (τ, σ) with τ in T and σ in Σ such that two pairs of the class which have the same first element have also the same second element ; e.g. the class of all pairs (time-interval τ, space-interval σ traversed during τ) is a mapping ϕ according to this definition.

A variable quantity with domain Σ is now defined to be a one-to-one mapping s of Σ into the real numbers; for the case used as an example above, $s\sigma$ might be the length of σ in feet; similarly, if $t\tau$ denotes the duration of τ in seconds, t is a variable quantity with domain T. If s and t are variables with domains Σ and T respectively, and if ϕ is a mapping of T into Σ, then it is not hard to show the existence of a function f (which is of course just a mapping of the real numbers into themselves) such that $f(t\tau) = s(\phi(\tau))$. In the case used as an example, $f = 16I^2$, for $16I^2(t\tau) = s(\phi(\tau))$. This equality is usually written $s = 16t^2$; but one ought rather to write $s_t = 16I^2$ and to say "the function relating s to t is $16I^2$". The symbol $\frac{ds}{dt}$, which has not hitherto been used, is now introduced as a synonym for Ds_t: it is a pure function (a mapping of the real numbers into themselves) and is in fact the function which relates the instantaneous velocity v (which is a variable quantity) to the variable quantity t. If u and v are any two variable quantities, related by the function u_v, then the symbol $\int_a^b u\,dv$ is an alternative for $\int_a^b u_v$ and has in certain cases a physical interpretation.

This is a very drastic simplification—probably an over-simplification—of the author's views; but perhaps enough has been said to convince the reader that these views are worthy of close examination. They can be applied not only to physics, but also in co-ordinate geometry to the problem of tangents; in this field they lead to a much more satisfactory treatment than any given in the conventional text books.

The variable quantities which the author defines are not at all the same things as Weierstrass variables, which are symbols representing arbitrary elements of sets, or arbitrary real numbers. It is the tragedy of the conventional treatment of the calculus that these two notions have not been clearly distinguished; the author has done a great service in pointing out and resolving the confusion.

The reviewer wishes now to report on an experiment carried out in teaching calculus by the new method to a class of 80, made up of Arts, Science and Engineering students, some of whom had already studied calculus at school. The conditions for the experiment were by no means ideal: the differing mathematical background of the students, their varying reasons for taking the course, and their natural conservatism were all working against it, as was the requirement that the final examination paper be expressed in the traditional notation and terminology. There was definite hostility to the new method from a small group of Science students who had "done" calculus at school, had acquired a certain facility in the techniques of integration and differentiation, and had no interest in understanding the underlying principles: this reviewer at least has no sympathy with such students. Another small group of students, who were dissatisfied with their conventional school course in calculus, found that the new course cleared up most of their difficulties, and confessed themselves converted. The great majority of the students showed much the same response to the modern treatment as their predecessors have done to the classical treatment, and in the tutorial classes they appeared to have no greater difficulty in working out examples. It is unfortunate that there is as yet no convenient textbook to supplement the lectures and clear up misunderstandings; for the book under review is too large and too expensive for the average British undergraduate, and in any case it is too much taken up with polemics against the classical approach. The lack of such a textbook is less of a misfortune for American than for British undergraduates, for, in the reviewer's experience, the former are much less inhibited about bringing their difficulties to their teachers—they are also, incidentally, less conservative.

The reviewer's verdict, as the reader will have gathered, is definitely in favour of this book. Every teacher of the calculus must read it. And he will be thick-skinned indeed who, having read it, does not seriously rethink his lectures; the thinner-skinned reader will rewrite his—and the reviewer confidently believes he will rewrite them à la Menger.

IAIN T. ADAMSON.

Geometrical Mechanics and de Broglie Waves. By J. L. SYNGE. Pp. vii, 167. 25s. 1954. Cambridge monographs on mechanics. (Cambridge University Press.)

In the space-time (Minkowski M_4) of special relativity let $\mathbf{x}$ be a point (event) and $\boldsymbol{\alpha}$ a unit vector. Suppose a function (medium function) $f(\mathbf{x},\boldsymbol{\alpha})$ to be defined for all $\mathbf{x}$ in some region M of M_4 and for all time-like directions $\boldsymbol{\alpha}$; let the fact of $\boldsymbol{\alpha}$ being a unit vector be used to make f positive homogeneous of degree unity in the components α_r of $\boldsymbol{\alpha}$. The curves (world-lines) in M defined by $\delta\int f(\mathbf{x}, d\mathbf{x}/ds)ds = 0$, where ds is the element of interval (Minkowski distance) will be called the *rays* for the given f. The geometry of these rays, which is the geometrical description of results in the calculus of variations, can be developed in complete analogy with Hamilton's geometrical optics in euclidean 3-space. In particular, with any given direction $\boldsymbol{\alpha}$ at a point $\mathbf{x}$ there is associated a slowness vector $\boldsymbol{\sigma}$ whose components are defined by $\sigma_r = -\partial f/\partial\alpha_r$ $(r = 1, 2, 3, 4)$. The derivatives of f being homogeneous of degree zero in the α_r can be eliminated from these equations to give a relation $\Omega(\boldsymbol{\sigma}, \mathbf{x}) = 0$ satisfied by all the vectors $\boldsymbol{\sigma}$ at $\mathbf{x}$. If $\boldsymbol{\sigma}$ satisfies this relation, a corresponding $\boldsymbol{\alpha}$ can be found. Thus, if a 3-surface Σ_0 is given such that at any point $(P')\mathbf{x}'$ of Σ_0 a vector $\boldsymbol{\sigma}'$ exists which is normal to Σ_0 and satisfies $\Omega(\boldsymbol{\sigma}', \mathbf{x}') = 0$, then, if $\boldsymbol{\alpha}'$ is the corresponding $\boldsymbol{\alpha}$, there exists a congruence of rays such that the member through $\mathbf{x}'$ has direction $\boldsymbol{\alpha}'$. A point P can then be found on this member such that the action $\int_{P'}^{P} f\,ds$ along the ray has a given constant value. As P' varies in Σ_0, P generates another 3-surface Σ. Taking different values of the constant, we obtain a family of surfaces Σ which can be generated by taking any one of them to be Σ_0. Synge calls these surfaces *de Broglie waves* in M_4.

The rays may be taken to be the world-lines of particles moving in a field of force which determines the medium function. A kinematic picture of their motion can be got in the usual way of special relativity in terms of the experience of an observer in the space-time. Also, any one of the 3-surfaces Σ can be interpreted as the history of a moving 2-surface in the experience of this observer. *The relationships between the motions of the particles and the associated de Broglie waves are found to be in fact the same as the familiar ones in wave mechanics.* This remarkable discovery by Professor Synge is the central feature of his important book. He has found that "the de Broglie waves had been in (mathematical) existence since the time of Hamilton".

The setting of the work in its background is described in the first chapter and the general theory as outlined above, together with its application to various particular problems, is developed in the next two chapters.

Up to this stage the de Broglie waves have no phase or frequency (or amplitude). This is in precise analogy with the corresponding theory in geometrical optics. The assignment of these quantities in the present theory constitutes the process of *quantization*. Professor Synge makes an illuminating comparison between this step and the passage from geometrical to physical optics. In two simple cases he establishes an exact connexion between his results and those of standard quantum wave-mechanics. He then introduces a universal process of *primitive quantization* which is simply that two de Broglie waves

of a family (Σ) are to have equal phase if the constant of action along the associated rays is an integral multiple of Planck's quantum of action h. This process is suggested by the simple cases mentioned, although it does not agree quite exactly with one of them. Nevertheless, as Professor Synge shows, it leads to the usual results for the fine-structure of the energy-levels of hydrogen and for the simple Zeeman effect for this atom. Thus it has still to be decided whether the process has a status similar to that of some of the rules for obtaining good approximate results in physical optics or whether it possesses a more fundamental validity.

The final chapter deals with " some generalizations " of the work to manifolds of N dimensions and so, in particular, to the treatment of systems of two interacting particles.

Professor Synge's purpose in the present book is to develope a " coherent mathematical theory ". This he does with the elegance and clarity that characterize all his writings. He assists the reader with exceedingly instructive diagrams. It is greatly to be hoped that he and others will pursue what may prove to be a revolutionary approach to quantum mechanics.

W. H. McCrea.

An Analytical Calculus. III. By E. A. Maxwell. Pp. vii, 195. 15s. 1954. (Cambridge University Press)

The greater part of this volume is devoted to functions of more than one variable. It is unusual to find a textbook at this level which gives so much space to this subject, and the clear exposition to be found here is likely to prove popular with university students and scholarship candidates. The scope of the book is roughly that of a first year university course; there are chapters on maxima and minima (in which Lagrange's method of undetermined multipliers is clearly explained) and on Jacobians.

As in the preceding volumes, the standard of rigour is sensible, and the assumptions made are carefully stated. The formulae of partial differentiation are derived on the assumption of the continuity of the partial derivatives, but later on the author takes pains to show that the " negligible " terms in certain multiple summations do in fact tend to zero in the limit. Dr. Maxwell never conceals the fact that there is a pill to be swallowed, but he provides a coating of sugar by introducing many topics with particular examples and by referring to geometrical illustrations where appropriate.

The rest of the book deals with the sketching of curves. This is an exciting subject, full of interest for the budding mathematician, and for this reason, if for no other, most teachers will wish to place it earlier in the course than this book suggests. There is, however, very little in the treatment given here which the pupil could not tackle at a much earlier stage. The subject is one which suffers from being on the fringe of several branches of mathematics. The approach of this book is purely analytical, and the teacher will probably wish to supplement it with reference to the methods of algebraic geometry.

The method described for finding successive approximations at the origin and at infinity is that of undetermined coefficients and exponents. This is quite general, though some pupils may find it rather difficult and tedious to apply in simple cases where other methods are available.

It is a pity that both the more substantial examples given of change of variable in partial differentiation employ a special method applicable only to the form $\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2$. It is not clear on p. 173 why the stipulation is made that the factors should be all distinct.

This volume can be recommended as heartily as its predecessors. It will be all the more welcome because its subject-matter receives scant attention from many existing textbooks.

D. A. Q.

Singular Integral Equations. By N. I. MUSKHELISHVILI. Translated from the 2nd Russian edition by J. R. M. Radok. Pp. 447. Fl. 28·50. 1953. (Noordhoff, Groningen)

The present book is a translation of the second Russian edition, which was published in 1946. The singular integral equations mentioned in the title are of the form

$$A(s)\phi(s)+\frac{1}{\pi i}\int_L \frac{K(s,t)\phi(t)\,dt}{t-s}=f(s), \qquad \ldots\ldots(1)$$

where L consists of a finite number of non-intersecting, piecewise smooth, open or closed simple curves in the finite part of the complex plane, and the integral is to be interpreted in the Cauchy principal value sense. The functions $A(s)$, $f(s)$ and $K(s,t)$ are required to satisfy Hölder conditions on L, and the solution $\phi(t)$ is desired to satisfy a similar condition.

Such equations arise in a very natural way in many problems of mathematical physics; they were encountered by Poincaré in tidal theory and by Hilbert in connection with the Dirichlet problem for plane domains.

The first step in solving (1) is to write it in the form

$$A(s)\phi(s)+\frac{B(s)}{\pi i}\int_L \frac{\phi(t)\,dt}{t-s}+\frac{1}{\pi i}\int_L k(s,t)\phi(t)\,dt=f(s),$$

where $B(s)=K(s,s)$ and

$$k(s,t)=\frac{K(s,t)-K(s,s)}{t-s}.$$

The function $k(s,t)$ is then an ordinary Fredholm kernel; neglecting it for the moment, we concentrate our attention on the "dominant equation" associated with (1), namely,

$$A(s)\phi(s)+\frac{B(s)}{\pi i}\int_L \frac{\phi(t)\,dt}{t-s}=f(t). \qquad \ldots\ldots(2)$$

Once a general solution of (2) has been obtained, it is possible to use it to reduce (1) to an ordinary Fredholm equation, which can then be handled by classical methods.

In order to solve the dominant equation the author reduces it to the solution of what he calls the Hilbert problem: this is the problem of finding a function $\Phi(z)$, regular in the complex plane except on L and behaving like a polynomial at infinity, having boundary values $\Phi_+(t)$ and $\Phi_-(t)$ on the two sides of L, and satisfying the equation

$$\Phi_+(t)=G(t)\Phi_-(t)+g(t), \qquad \ldots\ldots(3)$$

where $G(t)$ and $g(t)$ are given functions satisfying Hölder conditions on L, and $G(t)\neq 0$ on L. The last condition is equivalent to requiring that

$$[A(t)]^2-[B(t)]^2\neq 0$$

for t on L. New difficulties appear if this is not satisfied.

The solution of the Hilbert problem depends essentially on the Plemelj formulae, which are as follows: let $\phi(t)$ satisfy a Hölder condition on L, and let

$$\Phi(z)=\frac{1}{2\pi i}\int_L \frac{\phi(t)\,dt}{t-z}$$

when z is not on L. Then $\Phi(z)$ is regular except on L, and its boundary values $\Phi_+(s)$ and $\Phi_-(s)$ on the two sides of L satisfy the relations

$$\Phi_+(s) - \Phi_-(s) = \phi(s),$$

$$\Phi_+(s) + \Phi_-(s) = \frac{1}{\pi i}\int_L \frac{\phi(t)\,dt}{t-s},$$

the integral being a Cauchy principal value.

The method whose basic principles we have outlined above was first introduced by Carleman in solving a particular type of singular integral equation; its extension to more general types of equation and systems of equations and the technical details of its application in various circumstances have largely been the work of a school of mathematicians centred at Tbilisi (Tiflis), the capital of Georgia, among whom we may mention N.I. Muskhelishvili, S. G. Mikhlin and I. N. Vekua. The present translation, a preliminary duplicated form of which was issued by the Australian Department of Supply and Development in 1949, makes a full account of this work available to those unable to read Russian or to obtain access to the original literature.

A number of applications of the results are discussed in the present book; among them are the Dirichlet problem for plane domains and some generalisations of it, some problems from elasticity theory and the well known integro-differential equation that arises in the theory of aircraft wings of finite span. These are only a small selection of the many problems to which these methods are applicable; many others have been discussed in the works of the author and his collaborators, and reference may also be made to the recent translation of the author's book *On basic problems of the mathematical theory of elasticity* (Groningen, 1953).

F. SMITHIES.

Linear Analysis. By A. C. ZAANEN. Pp. vii, 600. 76s. 1953. (North-Holland Publishing Co., Amsterdam; Noordhoff, Groningen)

This book has the subtitle "Measure and Integral, Banach and Hilbert Space, Linear Integral Equations". It gives an account of all these subjects which is within the capacity of anyone equipped with the mathematics of the conventional Honours course, and with the requisite interest.

The first part of the book deals with the theory of Lebesgue Measure and Integration, beginning with the elements of point-set theory and developing both the theory of ordinary integration and that of additive set functions on general sets, along the lines which Carathéodory made classical. The L^p spaces are introduced, and their generalizations, the Orlicz spaces, are also treated. It is an unusual feature of the book that in all applications to specific function spaces the Orlicz spaces are considered in detail.

The second part gives the theory of Banach Spaces and of Hilbert space, and of the theory of bounded and compact operators in these spaces. The theory of normal, unitary and selfadjoint operators in Hilbert space is discussed, but the standard resolution theorems for these operators are not given. On the other hand, the theory of general compact operators is developed in great detail, and a very good account is given of the Fredholm theory and of the Riesz-Schauder Theory.

The third part of the book applies the theory to nonsingular linear integral equations, and in particular developes the theory of integral equations, with normal and symmetrizable kernels, in considerable detail.

All chapters of the book are provided with well-chosen sets of exercises, and the exposition is admirably suited to reading by a student. Considered as an introduction to its subject the book can in general be very strongly recom-

mended. It has, however, some minor defects. One of these is the lack of any discussion of one of the most important applications of the theory, to differential equations ; but this follows from the author's restriction of his discussion to bounded operators, and can readily be remedied by reference to numerous texts, whereas the applications he has discussed are not as well known. A second point is that the notation and terminology sometimes differ from that in current use ; for example, the older notation for sum and product of sets, and the old definition of a compact set are used ; and the author calls an incomplete normed space an incomplete Banach space, which in the standard usage (though of course not in his) would be a contradiction in terms.

An objection of a perhaps controversial nature is to the fact that the proof of the Hahn-Banach Extension theorem, which is probably the most important single theorem in linear analysis, is regarded with suspicion by the author. A space which has the property which, if this theorem is accepted, holds for all linear convex spaces, is said to have the property (Ext) and where necessary it is made part of the hypothesis of various theorems that a space should have this property. The reason that the theorem is not used is stated to be that " it is founded upon an axiom about the truth of which there exist controversial opinions, depending upon one's attitude to the foundations of mathematics ". This concession to intuitionism or constructivism might be held, depending upon one's attitude to these doctrines, to have some point if it were part of a general attempt to see how much of the theory of linear spaces could be expounded in constructivist terms. The reviewer's guess is that the answer would be very little. However, the axiom in question, which is surely the axiom of choice, is used tacitly by the author in a number of proofs ; in some of these its use could be avoided, but in others it could not. Altogether it seems simpler to accept intuitionist criticism than to attempt occasional appeasement : but no loss to the book's content results from this compromise.

The book is well and carefully written, and excellently printed and produced.

J. L. B. COOPER.

Eléments de Mathématique. By N. BOURBAKI. XV, Première Partie, Les Structures Fondamentales de L'Analyse. Livre V, Espaces Vectoriels Topologiques, Chaps. I & II. Pp. 123. 1500 fr. 1953. Actualités Scientifiques et Industrielles, 1189. (Hermann, Paris)

The two chapters of this book are entitled respectively " Topological vector spaces over a field with valuation " and " Convex sets and locally convex spaces ". In accordance with the general method of the treatise of Bourbaki, which is to go from the general to the particular, these chapters commence the study of linear topological spaces by considering the most general types which are of importance in mathematics. The first of them studies spaces on arbitrary valued fields ; in the second chapter the field is restricted to be the real field, for the most part, and the complex field in some sections. Nevertheless, these chapters are filled with a great many of the major theorems which are of interest to those who are not concerned with these general fields. The first chapter, for example, deals with normed spaces and with spaces which are like Banach spaces except that the field over which they are defined may be different ; and it turns out that a remarkable number of the important theorems proved by Banach for his spaces remain true for the more general case.

The second chapter is concerned with locally convex spaces, particularly over the real field. It is of a more geometrical nature ; the Hahn-Banach theorem is proved in an illuminating geometrical form as well as in an analytical form. Other subjects discussed are the definition of a topology by means of pseudonorms, the properties of convex sets and in particular the important

theorem of Krein and Milman on extremal points, and an interesting fixed point theorem of Markhoff and Kakutani.

All sections are complemented by numerous exercises, which serve to give accounts of further developments of the theory. It is superfluous to recommend this book to those acquainted with other works of the series : but it should be said that an acquaintance with the first two and the ninth chapters of the book on Topology are more or less essential prerequisites for anyone reading it.

J. L. B. COOPER.

Symbolic Operators. By J. P. DALTON. Pp. xvi, 194. 30s. 1954. (Witwatersrand University Press, Johannesburg)

The purpose of this book is to give an account of the symbolic methods of the operational calculus, in the form in which it was used by Heaviside, making use of the symbolism itself and eschewing outside aids such as the Laplace Transform.

The author works with a field of functions of the form $t^{\delta-1}f(t)$, where $f(t)$ is continuous in a closed interval $(0, T)$, and $\delta>0$, and works in terms of the operator Q, where $Qu=\int_0^t u(y)dy$. The basic difficulties of the calculus are traced to the fact that Q has no inverse which turns admissible functions into admissible functions. To get round this difficulty the author works with what, in linear analysis generally, is called the resolvent operator of Q, namely the operator $(1-\lambda Q)^{-1}$. By expressing other operators of the calculus in terms of this operator, he is able to prove the formulae of the theory of ordinary differential equations, including the expansion theorem (which he attributes to Heaviside but which was in fact fully proved by Cauchy) and he also develops the theory of differentiation of fractional order.

To discuss partial differential operators, operators similar to $(1-\lambda Q)^{-1}$ are defined in which the λ is replaced by a differential operator. Using this technique initial value problems for hyperbolic and parabolic differential equations are discussed.

The book constitutes an attempt, and a pretty successful and ingenious one, to justify processes of the symbolic calculus by methods lying within the framework of classical analysis. The value of this attempt, in the state of the subject today, is rather difficult to assess.

For one thing, the exposition is frequently very heavy. The author relies on methods involving expansions in series, and many of his arguments could have been considerably condensed if he had relied on expressions for operators as integrals rather than series. At the best, it remains doubtful whether arguments based on the techniques of the classical Riemann integral theory of real variables could compete in shortness with those using Laplace or Fourier transforms and the powerful methods of the complex variable theory.

In the second place, the problem is very much older than the author appears to think ; he quotes " a recent claim that ' Bromwich was the first to explain, and to a certain extent justify, Heaviside's methods ' " and states that it is " at least disputable since a verification of Heaviside's technique by an alternative and essentially different procedure neither explains his technique nor establishes its validity ". He goes on to say that the Heaviside and Laplace Transform techniques differ in their mathematical techniques and also because the latter involves integration over an infinite interval. Now the statement quoted is mainly disputable because Cauchy, rather than Bromwich, was the first to explain and to some extent justify these techniques. It is, moreover, hard to see how a technique which is thought dubious can be justified save by showing that the results it gives agree with those given by

wellfounded methods: Professor Dalton himself draws very largely on the techniques of the classical theory of differential equations to justify the results of symbolic processes. The objection to the use of the Laplace Transform is very closely analogous to the objections by some geometers to the use of analytic methods; and its advantages are the same as those which make other geometers prefer analytic tools, namely that one can go very much further into the subject with very much less trouble. The objection that the Laplace Transform methods involve integration over an infinite interval are easily overcome in all the cases where it is relevant—that is, precisely those where the other technique does not involve integration over an infinite interval. As against this, the Laplace Transform method does give precise conditions for uniqueness of solution of equations which are not obtainable from the symbolic technique.

The problem of a direct justification of the symbolic calculus is, nevertheless, of great mathematical interest. It seems to me likely that it cannot be carried out without considerable restrictions on generality unless the field of operands is extended to include objects more general than functions: and indeed in considering partial equations the author is forced by his methods to restrict attention to analytic solutions. From a pedagogical point of view, it does seem highly desirable that the direct operational methods should be taught more often and with more rigour; and the book will be valuable if it encourages this, and is useful as a guide to methods of doing this.

The Witwatersrand University Press is to be congratulated on the excellent production of this, its first monograph on a mathematical subject.

J. L. B. COOPER

Topologie. II. By CASIMIR KURATOWSKI. 2nd edition. Pp. viii, 443. $6. 1952. Monografje Matematyczne, 21 (Polska Akademia, Warsaw)

The manuscript of the present volume was practically complete in September 1939. Fortunately it was preserved from destruction by the Nazis and the author was able to revise it for publication in 1950. He decided that to include a treatment of the most recent developments in algebraic topology would cause too great a further delay in publication, so that the volume as published is much as it would have been had it appeared 13 years earlier; new additions are in the nature of recent refinements of theories already well developed by 1939.

This explanation of the long gap between the appearance of the first edition of Volume I and that of the present volume is given by the author in his preface; it should not be taken to imply that the present volume is, in any significant sense, out of date. On the contrary, it is an invaluable and masterly work, as one would expect of its illustrious author. It forms, with Volume I, a comprehensive study of the fundamental results of the analytical topology of metric spaces which, roughly speaking, are provable without the aid of homology theory. The chapters of Vol. II, following on those of the second edition of Vol. I, are numbered 4–9. Chapter 4, on compact spaces, discusses the space of subsets of a given metric space with

$$\rho(A, B) = \max\{\sup_{a \in A} \rho(a, B), \sup_{b \in B} \rho(b, A)\};$$

the space Y^X of maps $X \to Y$, X compact, Y metric, with

$$\rho(f, g) = \sup_{x \in X} \rho(fx, gx);$$

upper and lower semi-continuity and semi-continuous decompositions; and further properties of dimension. Chapter 5, on connectedness, deals with the

separation of connected sets, unicoherence, the theory of continua and irreducible and indecomposable spaces. Chapter 6 is on locally connected spaces and Chapter 7, on absolute retracts, discusses extensions of maps and homotopies and contractible spaces. Chapter 8 deals with the theory of mappings of spaces into abelian groups (foretaste of cohomology!) developed by the author and S. Eilenberg, and the final chapter concerns the topology of the plane with several applications to analysis.

Since this review appears rather a long time after the appearance of the book, further remarks on the treatment and the lay-out of the material are probably superfluous; the above list of contents, together with the reputation of the author, should be sufficient commendation. Perhaps it would be fair to say that this volume is more a work for specialists than Vol. I, but it should certainly become a standard work in this field.

P. J. HILTON

Grundlagen der analytischen Topologie. By G. NÖBELING. Pp. x, 221. DM 33; linen, DM 36.60. 1954. Grundlehren der mathematischen Wissenschaften, 72 (Springer, Berlin).

This, the latest volume of a famous series, properly claims to lay the foundations of analytical topology, and the fact that this is achieved in only 220 pages is itself a commendation. The material of this book, in so far as it concerns topological spaces, is largely co-extensive with that of N. Bourbaki, *Topologie générale* (Ch. I, Structures topologiques; Ch. II, Structures uniformes). That is to say, after an introductory chapter of set-theoretical preliminaries, the second chapter, comprising more than half the book, discusses metric and quasi-metric spaces; closure, interior and adherence of subsets and limit points of filters; the induced topology of a subset; continuous maps and homeomorphisms; separation axioms; compactness; denseness; connectedness; topology of factor spaces; product spaces; and the Cech and Wallman compactification of a T_1-space. The third chapter discusses uniform structures on sets and induced topologies, and generalises familiar theorems on uniform continuity and uniform convergence in metric spaces.

However, this book is far from being just a German guide to Bourbaki. For the author, following Wallman and others, does not regard the topological space as the basic object of study of the analytical topologist, but rather the topologised ordered set (or partially ordered set in the older terminology). A topology T on an ordered set, $\mathcal{U}$, is an endomorphism $T : \mathcal{U} \to \mathcal{U}$ such that

$$\text{(i)}\quad A_1 \leqslant A_2 \to TA_1 \leqslant TA_2;$$

$$\text{(ii)}\quad A \leqslant TA;$$

$$\text{(iii)}\quad TA = TTA.$$

(If $\mathcal{U}$ is the set of subsets of a topological space, then TA may be taken as the closure of A.) In the class of ordered sets appears the subclass of lattices, and the author defines a topological space as a topological atomic Boolean complete lattice. That is, a topological lattice S is a topological space if it has the following properties:

(*a*) S is distributive and complemented (Boolean);

(*b*) every element of S is expressible as $\bigcup P_i$, where the P_i are atoms of S, that is, elements such that

$$O < Q \leqslant P_i \to Q = P_i \text{ (atomic)};$$

(c) given any collection of elements $A_i \in S$, then $\bigcup_i A_i \in S$ (complete).

Moreover, S is said to be a classical topological space if we also have

(iv) $$T(A_1 \cup A_2) = TA_1 \cup TA_2.$$

One may recognise that if Σ is a topological space in the usual sense, then the set of its subsets is a classical topological space S; conversely, the atoms of a classical topological space, S, are the points of a topological space, Σ, in the usual sense, with $\overline{A} = TA$, where T is the topology on S. The author calls Σ the carrier of S. This point of view, and the generality in which the author states and proves many of his results, make the book difficult to read, and the reviewer would have appreciated the use of some system, such as asterisking, to draw the reader's attention to the theorems of "classical" topology. However, the extra material on topological and uniform lattices fully justifies the book's publication, and it is difficult to imagine how the contents could have been presented with greater clarity. The author has demonstrated in a particularly satisfactory way how the Weil and Bourbaki definitions (in terms of neighbourhoods of the diagonal in the product space) are related.

The book is well provided with examples and exercises; their abstract nature is a natural consequence of the fundamental nature of the material and the absence of any treatment of such applications as the theory of curves, dimension theory, or combinatorial topology. In this respect, the less fundamental book of M. H. A. Newman, *Topology of plane sets*, gives the reader a better picture of the role of analytical topology within the general mathematical framework; of course, the two books have quite different motivations.

The reviewer noticed the following misprints:

on p. 10, l. 9, $\bigcap_c A_i$ should be $\bigcup_c A_i$;

on p. 45, l. 11, $\overline{A}_1 \cap A_2$ should be $\overline{A}_1 \cup A_2$;

on p. 52, l. 1, Σx_n^{-2} should be Σx_n^2.

He is also puzzled to know how to translate the two words "Gitter" and "Soma". The Gitter of a filter-base R is the set of all sets A such that $A \cap B$ is non-empty for all $B \in R$. A Soma of an ordered set is just an element of that set; perhaps "body" would be an acceptable translation in this case.

P. J. HILTON

Principles of Mathematical Analysis. By WALTER RUDIN. Pp. x, 227. 40s. 1953. (McGraw-Hill)

Theory of Functions of Real Variable. By HENRY P. THIELMAN. Pp. xiv, 209. 35s. 1953. (Butterworth Scientific Publications, London)

These two books on analysis are designed, it is presumed, for the same type of student, viz., one who has completed the usual manipulative course on calculus and is ready for a more logical approach to function theory. Both books are essentially on the real variable (although in Rudin's book complex numbers and complex-valued sequences and series are discussed), but they are very different in style and emphasis. Thielman discusses many general concepts in point set topology, giving an introduction to Hausdorff, topological, metric and linear spaces, while Rudin is more limited in his scope, but his book can be recommended as a well-expounded and carefully planned text.

Rudin's book begins with a good chapter on the real and complex number systems, followed by a chapter on the elements of set theory. Here one might criticise the rather abrupt manner in which he introduces the fundamental ideas; neighbourhoods, limit points, closed, compact, perfect and dense sets

all appear in one short paragraph. However, sufficient set theory is given to be a foundation for the excellent chapters on sequences, series, continuity, differentiation, etc., which follow. There is a full discussion of the Riemann-Stieltjes integral (somewhat impaired by a serious blemish in § 6. 11), the Riemann integral being treated merely as a special case, while the last chapter is a treatment on general lines of the Lebesgue theory of measure and integration. There are adequate sets of well-chosen examples at the ends of all the chapters and care has been taken to provide hints where necessary.

Almost half of Thielman's book is devoted to chapters on the real number system, point sets, spaces and sequences, and these ideas are then used in the treatment of limits and continuity of functions. There is a rather elementary and sketchy chapter on infinite series which seems a little out of place in the general scope of the rest of the book. The last part of the work consists of three chapters on measure, integrals, the indefinite integral and derivatives. Problems are included in the body of the book and the author claims that no more theory is usually required than that which precedes the problem. This may be true, but quite important results are given as problems and in many cases the average student would find them difficult to solve without help. It would appear that the proof reading has not been very carefully done. A first reading revealed over fifty misprints, including wrong references to earlier parts of the book. While many of the errors are trivial there are some which might prove worrying to a student: e.g., $\leqslant$ for $\geqslant$ on p. 22, " less " for " greater " on p. 90, an error (or misprint) in the proof of Theorem 6.2.1 on p. 98, $\delta/2$ for δ on p. 105, errors in the proof of Theorem 7.4.2. on pp. 113–114, and the misprints in the definition of the Riemann-Stieltjes integral on p. 186.

R. P. GILLESPIE

Introduction to Modern Algebra and Matrix Theory. By R. A. BEAUMONT and R. W. BALL. Pp. xiv, 331. 30s. 1954 (Rinehart & Co., New York; Constable, London)

This is a textbook for Honours students in Mathematics, and should also prove helpful to research workers, in other fields, who wish to understand the algebraic techniques they use.

Matrices with real elements are introduced before the field axioms are stated, so that the student may better appreciate the significance of these axioms; similarly, transformation groups prepare the reader for abstract group theory. Such basic topics as determinants, vector spaces, rings, homomorphism, polynomial algebra, factorization theory and field extension are thoroughly discussed, special attention being given to ordered integral domains and to canonical forms of matrices under various groups of transformations.

The arrangement of material is carefully planned, the explanations and motivations being admirably clear, though minor ambiguities and obscurities have not been entirely avoided. Thus Ex. (52.1) seems absurd at first sight, while the use of the word " unnecessary " in Ex. (52.4) could mislead; the proof of Corollary (64.4) is incomplete, the remark needed to complete it only being given (as an exercise) two pages later, without comment. The reviewer, while not holding the authors responsible for what has become a widespread habit, would like to register his objection against the usage (cf. the assertion $-(-b)=b \in E$ on p. 179, and the second displayed formula on p. 273) of such a phrase as $A=B=C$, standing for the statement " $A=B$ and $A=C$, whence $B=C$ "; this statement is, surely, better expressed symbolically as $B=A=C$ (which can be read at a glance from left to right, and readily extends to statements involving three or more connectives).

Undisplayed formulae are run on indiscriminately from one line to the next, italic type is not used in the statements of theorems, and the use of the solidus

occasionally offends against standard practice ; the typography is otherwise adequate. The reviewer noticed no misprints. There is a good index, and, for the convenience of the teacher, every statement not proved (in the fullest possible detail) in the text is set out formally as a numbered exercise ; the other exercises (all of a rather routine type) are well chosen.

Some of the more mundane algebraic concepts and techniques covered in the degree courses of most British universities find no place in this book. However, the authors' understanding of the novice's difficulties has enabled them to provide a sure stepping-stone for the student who wants to learn more about Abstract Algebra.

M. P. DRAZIN

Theory of Equations. By C. C. MACDUFFEE. Pp. vii, 120. 30s. 1954 (John Wiley, New York ; Chapman & Hall, London)

This little book has grown from a course the author has given repeatedly for students in their junior and senior, that is, third and fourth, year. It is (" therefore "?) quite suitable for our First Year Honours students. It covers the bare essentials of the theory of polynomials in one and several indeterminates, and the seven chapters deal successively with Linear Equations, Rational Solutions, Polynomials, Real Roots, Complex Roots, Relations among the Roots, Systems of Higher Degree. A small amount of abstract algebra is introduced in order to give the statements greater generality. An interesting and attractive feature is that the fundamental concepts of algebra are not treated systematically, but as and when the occasion to use them arises naturally. For example : systems of linear equations lead to the concept of an equivalence relation ; manipulation of polynomials to rings and fields ; equations of higher degree to euclidean rings and unique factorisation domains. The treatment is confined to the most elementary aspects of the subject matter and avoids all deeper theorems. The title of the book is, therefore, a little misleading ; for it does not do justice to the classical, nor to the modern theory of equations. But it can be recommended as a first introduction to university algebra.

K. A. H.

An Introduction to Mathematical Thought. By E. R. STABLER. Pp. xiv, 268. $4.50. 1953. (Addison-Wesley Publishing Company, Cambridge, Mass.)

There are good things in this book. It is as well to underline this, for the fact is apparent only after an initial sense of disappointment is overcome. It would be a pity if the work were dismissed as yet another well-meaning and inadequate attempt on the vexed question of what is mathematics, and why. The work grew out of an elementary course at Hosfra College, and for an English public it might prove most valuable on reversing the process,—using the book as a source for talks and exercises for Sixth Forms. The title is in fact an unhappy shortened version of " introductory material for the dual study of the role of thought in mathematics and the role of mathematics in thought ", the phrase used in the introduction. As " introductory material ," the subject-matter presented makes no claim to being a finished highway, but only selected blocks for the building of one. The author has selected what he knew and liked best ; if many more people do the same, there is good hope that the highway may some day be truly built. And it would be an excellent and revealing action if in the final co-operative effort all acknowledged with the same scrupulous loyalty as Mr. Stabler the individual aid on which their finished work had depended. He shows an awareness of human realities also

in the care with which he generally provides symbols with pronounceable names, labels fallacies, and crosses out faulty work in the text, as well as continually reverting to practical reasoning in the exercises, and spurring interest by setting problems in anticipation of the text. One of the best practical exercises devised has been expanded to half a chapter on " The Nature of Mathematical Knowledge " (Chap. 1), and concerns modulo arithmetic and the change of number base, using the addition table for days of the week. The fact that this is the first thing seen in the shop-window and that there is nothing in the same proportion inside is one of the points likely to put off the casual reader. About half as much space allotted to the geometries and a quarter to other solved and unsolved problems make up a somewhat badly balanced basis for the conclusion as to the " relative nature of mathematical truth ". But it is well worth while ignoring this and other shortcomings, secondary to the avowed *exploratory* purpose (and largely recognised by the author himself), and to take what is provided as a skeleton, partly filled in, partly to be completed, after the manner of intelligence tests and exercises in logical reasoning. To correct all flaws and notational inconsistencies in a second edition would be a daunting task, as always; but the index could readily be perfected and would be a powerful aid in the process.

R. C. H. YOUNG

Linear Equations in Applied Mechanics. By H. F. P. PURDAY. Pp. xiv, 240. 20s. 1954. (Oliver & Boyd)

The overall plan on which this book is based is a sound one. Starting from the simplest possible linear equation $ax = c$ the author proceeds to discuss in turn sets of simultaneous equations, difference equations, ordinary and partial differential equations, concluding with a chapter on integral equations. Matrices are introduced at an early stage and used extensively throughout the book. Two chapters are devoted to vectors and tensors and their applications to rotation, heat flow, stress and strain and fluid motion.

The properties of simultaneous equations and determinants are discussed in the first chapter. A carefully devised explanation here is marred by the premature introduction of matrices. Teachers of mathematics will appreciate the virtue of introducing one new idea at a time; the main ideas, together with the various special cases that simultaneous equations provide, offer quite sufficient food for thought for the reader who is new to the subject. Matrices and their properties are in fact described with great care in the second chapter, which is probably the most useful in the book. Examples of these are provided in the third chapter by applications to beams with concentrated loads. There follows a rather scrappy chapter on sequences and differences. Features of this that call for critical comment include the introduction of differences without any explanation as to how they are formed and the absence of any mention of interpolation, in spite of the fact that three pages are devoted to subtabulation.

The next two chapters are concerned with ordinary differential equations. The first contains one or two standard methods and a discussion of the second order constant coefficient equation, with particular reference to vibration problems. Equations of higher order surprisingly receive no mention. The next chapter is devoted to non-constant coefficient equations, introducing the equations of Bessel and Legendre, Fourier series and general orthogonal functions. The chapters on scalar, vector and tensor fields follow. The presentation here is both sophisticated and condensed, neither of which feature is likely to be helpful to a beginner. The same is true of the remaining chapters, on partial differential equations and integral equations. To quote but one example, the author contrives to proceed from a definition of complex numbers to conformal mapping in exactly three pages.

It is clear that the author has a wide and sound knowledge of mathematics and its applications to the problems of engineering, but the teaching experience of the reviewer suggests that he has grossly overestimated the capacity of assimilation of the student who has no more than an elementary knowledge of the calculus (for whom, according to the preface, this book is written). The fact is that practically every topic introduced requires much more space for an adequate treatment than it is possible to allow in a book of this size. Moreover the reader will not be helped by the fact that much of the theory is presented in a formal abstract manner, often lacking in lucidity. The author rightly places great stress on numerical applications, but although in a number of examples the arithmetic is shewn in full, the processes used are often inadequately described. This is particularly true of the examples on relaxation methods. All in all, this would have been a much more useful book if the care lavished on say the second chapter had been applied to a much narrower range of topics, to the exclusion of all others.

Although the printer has done his best, the book has an unprofessional appearance in places due to the non-observance of some of the accepted conventions for mathematical manuscript. These faults include the use of multiplication and other symbols as abbreviations in the text and the excessive use of brackets, while the frequent appearance of matrices has produced in places some awkward arrangement of both text and displayed formulae, in order to save space. A number of minor misprints have come to light. It is unfortunate for publicity that one of these should have been reproduced on the dust cover!

B. M. BROWN

An Introduction to the Mathematics of Physics. By J. H. AVERY and M. NELKON. Pp. vii, 178. 9s. 6d. 1954. (Heinemann)

This book has been written so that the student of physics at the G.C. Advanced level may familiarise himself with the application of mathematics to the physical problems he will have to solve. That is the object the authors had in mind when producing the book and, because of the excellent way in which this object is achieved, the book will be welcomed by both student and teacher. The mathematical concepts are set out and explained in a clear concise manner and the variety of applications is very wide indeed. There is, however, little rigour involved in the text, but, since this is not intended as a textbook for mathematicians but rather as an aid to physicists, this is probably a desirable quality.

One disappointment is that the student is not shown the best method of determining a linear law. The use of the obsolete intercept and gradient method is not as accurate as that of simultaneous equations if only because frequently a larger scale can be used. At the beginning of the chapter on trigonometry, the reader suddenly has six trigonometrical ratios thrust upon him. Surely the sine and cosine would have been enough at this stage. Since the authors intended the calculus to precede the mechanics one would have thought it would have been quicker and more desirable to determine the four fundamental formulae by means of the calculus. The concept of what is a " maximum value " and what is a " minimum value " may seem self-explanatory to those of us familiar with their use, but I doubt if this is always so for the student. For that reason I think a little more explanation at that point would be welcome.

These small shortcomings aside, however, the book is an admirable one and the authors will probably find that they have contributed considerably to the easing of the burden for the student of physics.

G. W. HANNAN

Die Elemente der Mathematik. Edited by G. WOLFF. Vorstufe: Rechnen. Parts 1, 2, 3. Pp. 119, 104, 80. DM. 2.80, 2.80, 2.40. Mittelstufe: I. Arithmetik und Algebra. Pp. 303. DM. 6.30. II. Geometrie und ebene Trigonometrie. Pp. 300. DM. 6.20. Oberstufe: III. Algebra und Analysis. Pp. 352. DM. 6.80. IV. Analytische Geometrie, Vektorrechnung, darstellende und projektive Geometrie, sphärische Trigonometrie. Pp. 288. DM. 7.40. Erganzungsheft: I. Einführung in die Mengenlehre. Pp. 65. DM. 3.80. (Hermann Schroedel, Hannover)

This thorough-going series of textbooks, covering the whole of the school course in mathematics, from the elements of arithmetic to the beginnings of university work, bears the impress throughout of the director, Dr. George Wolff, well-known to many members of this Association. He has been efficiently supported by a notable band of assistants, but one can hardly be mistaken in attributing the versatility, the painstaking accuracy, the wide field of view, the contacts with the arts as well as with the utilitarian applications, the constant attention to the historic developments of the subject, primarily to the impulse supplied by the directing editor himself.

Since these books could not be used as texts for our own classes, their chief value to the English teacher lies in comparing them with our own textbooks. If space were available, a detailed comparative study would be of great value to our readers; perhaps too to Dr. Wolff, who is clearly anxious to make this set of books a model of good teaching style. In short compass, all that we can do is to note briefly some outstanding merits. The diagrams are admirable; clear, accurate, stimulating, particularly, as one might expect, in the geometry sections, and not without a touch of humour—an over-crowded tram-car illustrating two non-equivalent sets. There is no undue timidity about the introduction of abstract concepts, again particularly in the geometry sections. Examples for the student are plentiful, varied, and not too difficult. Portraits of mathematicians, with good biographical notes, are scattered generously throughout the volumes. There is a tone of humanism, an underlying consciousness that mathematics has been and is a major cultural force in Western civilisation.

The sections on the elements of the calculus seem to me to be the least successful. They have an out-moded air, and bear marks of hasty construction. For instance, I can not believe that the sharp ebullition of e as the limit of $(1 + 1/n)^n$ is the best that the careful teacher can do on this vexed topic; and the introductory work on limits of sequences and of functions seems confused and hastily, even carelessly, put together. At times, too, the geometry seems to be looking back rather than forward, but here forgiveness is easy in view of the lucidity of the explanations and the clarity and beauty of the multitude of diagrams.

Altogether, a noteworthy series, well-printed, well-produced, full of good things, meriting the attention of young teachers who are anxious to keep abreast of methods of teaching in other countries. T. A. A. B.

Mathematics for the Secondary School. By W. D. REEVE. Pp. xii, 547. 1954. (Henry Holt, New York)

For many years now W. D. Reeve has set the standard for the training of teachers of mathematics in the United States, and a multitude of his pupils are carrying his methods and his ideals into their own classrooms all over the Union. In this large, beautifully produced book, he has gathered together much of his earlier writing, stitched and fitted it to a great deal of new material and so made one comprehensive survey of mathematics for the secondary school. From the high principles on which he asserts the right of mathematics to rank as a major subject in the school curriculum down to the detail of

presenting a numerical illustration of the sine rule, from answering the question "Why should we teach mathematics?" to discussing what mathematics we should teach and how in detail we should teach it, there is hardly a point on which the young teacher in need of guidance will fail to find it. The detail, of course, is sometimes not immediately applicable to our British modes and circumstances, though it is interesting but not surprising to see in several places the influence of Sir Percy Nunn.

Reeve has the great gift of making his readers think for themselves; hardly a page but raises some interesting considerations. Here is an instance, taken more or less at random. European visitors say that American school education is superficial. The charge, Reeve says, is true; but it is the necessary price for equality of privilege in education, and this in turn is the only safe foundation for a democracy. But is a superficial education a safe foundation for democratic government? Perhaps the word "superficial" is an unfortunate choice; would it not be fairer to say that American education is as thorough as that in, for example, a French *lycée*, but does not go over so much ground? Yet even if we change "superficial" into "less extensive", and even if we are prepared to pass over an implication that "equality" and "democracy" are roughly synonymous, the question still remains: is the "sacrifice" of standards and thoroughness which the author appears to demand really necessary in order to "make the world safe for democracy"? Is there not a danger that mediocrity will be nourished, and the highly intelligent children, from whom we must draw those future leaders which even a democracy must have, will be starved and stunted? I do not instance this in order to pick a quarrel with the author, but to emphasise that his book, in addition to asking and answering a host of questions, will provoke the thoughtful reader to ask himself many other questions, some of which he will find answered, perhaps not explicitly, elsewhere in the text, and some to which he himself must try to find answers.

Many of us sadly recall that it is some fifteen or sixteen years since last we saw the distinguished author in this country, though we have had pleasure in meeting his pupils and his pupils' pupils. But if this book should be his *nunc dimittis*, it proves that he is still holding to his high ideals, that he has never spared pains in collecting knowledge of teaching methods, that he has never ceased to believe and proclaim that mathematics must be a pillar in the school curriculum and that the teacher of mathematics has a great and growing responsibility to the community. Nor is he afraid to add that the community would do well to recognise in fuller measure than hitherto its debt to the mathematical teacher.

T. A. A. B.

Premier Colloque sur les équations aux dérivées partielles. Pp. 132. 1400 fr. 1954. **Second Colloque sur les équations aux dérivées partielles.** Pp. 129. 1500 fr. 1955. (Thone, Liège; Masson, Paris)

Everyone who is interested in partial differential equations will enjoy these informative collections of papers. It is very encouraging to observe that Continental mathematicians, especially in Belgium and France, are paying every bit as much attention to the more theoretical aspects of the subject, sadly neglected in this country, as to boundary value problems.

D. H. P.